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Research Article

CONSISTENCY AND CONVERGENCES ANALYSIS OF MODIFIED HYBRID FORMULA FOR SPECIAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

Block hybrid method is adopted in this paper for the direct solution of second order ordinary differential equations of the form y'' = (x, y) the method is driven by collocation and interpolation of power series approximate solution to give a continuous hybrid linear multistep methods which is implemented in block method to drive the independent solution at selected grid points. The properties of the drive scheme were investigated and found to be zero=stable, consistent and convergent. The efficiency of the method was tested and found to compare favorably with the existing methods.

Keywords: Consistency, Convergence, Analysis, Modified Hybrid Formula, Special Second Order, Ordinary Differential Equation.

INTRODUCTION

The paper outline some of the important theory behind stiff computation and to direct users of numerical software to those codes which are most likely to be effective for their particular problem. In sciences and engineering, mathematical models are formulated to aid in the understanding of physical phenomena. The formulated model often yield an equation that contains the derivatives of the an unknown function. Such equation is referred to as differential equation. Interestingly differential equation arising from the modeling of physical phenomena often does have exact solutions. Hence the development of numerical methods to obtain approximate solution becomes necessary. To that extent, several numerical methods such as finite difference methods, finite element methods and finite volume methods among others, have been developed base on the nature and type of the differential equation to be solved. A differential equation can be classified into Ordinary Differential Equation (ODE'S), Partial Differential Equation (PDE'S), Stochastic Differential Equation (SDE'S), Impulsive Differential Equation (IDE'S), Delay Differential Equation (DDE'S) etc. Stuart and Humphries (1996). In recent time, the integration of Ordinary Differential Equation (ODE'S) is investigated using some kind of block methods. This paper discusses the formation of implicit linear multistep method (LMM) for numerical integration of special second order ODE's which arises frequently in the area of Science and engineering especially mechanical system, control theory and celestial mechanics, Y. Skwame, J. Sunday and J. Sabo (2018).

In this paper the system of special second order ODE's of the form.

$$y'' = f(x,y), \quad y(a) = y_0 \quad y'(a) = \eta_0, \quad x \in [a,b] \tag{2.1}$$

The paper carried out the derivation of the method where we consider two- step with a single off-grid point, through interpolation and collocation method approach. The details analysis of the method which include order, error constant, consistency, stability and convergences were done, some numerical example were consider in the paper. Y. Skwame, G. M. Kumleng and J. A. Bakari (2017), I. H. Umar, (2008), Y. Skwame, J. Sunday and J. Sabo, (2018), and J. Sabo, T. Y. Kyagya, A. A. Bumbur, (2018).

METHODOLOGY

Consider the following parameter specification.

$\sum_{j=0}^{2k} a_j(x^j), \; j = 0(1)6, \; k = 2, t = 2, \; m = 2.$ where t and m are the interpolation and collocation points respectively. Specifically $[x_n, x_{n+1}, x_{n+\frac{3}{2}}]$ as the interpolation and $[x_n \;\; x_{n+1}, \; x_{n+2}]$ as the collocation points. The approximate solution of (2.1) is of the form,

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots\dots\dots\dots\dots\dots \tag{3.1}$$

The collocation equation is of the form

$$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots\dots\dots\dots\dots\dots \tag{3.2}$$

Interpolating (3.1) at $X = x_n, x_{n+1}, x_{n+2}$, and collocation (3,2) at $X = x_n, x_{n+\frac{3}{2}}, x_{n+2}$ give the following system of equations

$$\begin{aligned}
&a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 = y_n \\
&a_0 + a_1 x_{n+1} + a_2 x^2{}_{n+1} + a_3 x^3{}_{n+1} + a_4 x^4{}_{n+1} + a_5 x^5{}_{n+1} = y_{n+1} \\
&a_0 + a_1 x_{n+\frac{3}{2}} + a_2 x^2{}_{n+\frac{3}{2}} + a_3 x^3{}_{n+\frac{3}{2}} + a_4 x^4{}_{n+\frac{3}{2}} + a_5 x^5{}_{n+\frac{3}{2}} = y_{n+\frac{3}{2}} \\
&0 \quad 0 \quad 2a_2 + 6a_3 x_n + 12a_4 x^2{}_n + 20a_5 x^3{}_n = f_n \\
&0 \quad 0 \quad 2a_2 + 6a_3 x_{n+\frac{3}{2}} + 12a_4 x^2{}_{n+\frac{3}{2}} + 20a_5 x^3{}_{n+\frac{3}{2}} = f_{n+\frac{3}{2}} \\
&0 \quad 0 \quad 2a_2 + 6a_3 x_{n+2} + 12a_4 x^2{}_{n+2} + 20a_5 x^3{}_{n+2} = f_{n+2}
\end{aligned} \tag{3.3}$$

writing (3.3) in matrix equation we have a system of equation in the form

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+\frac{3}{2}} & x^2_{n+\frac{3}{2}} & x^3_{n+\frac{3}{2}} & x^4_{n+\frac{3}{2}} & x^5_{n+\frac{3}{2}} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x^2_{n+\frac{3}{2}} & 20x^3_{n+\frac{3}{2}} \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ f_n \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \tag{3.4}$$

Which can be solved by matrix inversion technique to obtained the values of parameters $a_j{}^s, \quad j = 0, 1, 2, \dots\dots\dots k+2.$
Thus the proposed continuous scheme is of the form

$$y(x) = \alpha_0(x) y_n + \alpha_1(x) y_{n+1} + \alpha_{\frac{3}{2}}(x) y_{n+\frac{3}{2}} + h^2 \left[\beta_0(x) f_n + \beta_1(x) f_{n+1} + \beta_2(x) f_{n+2}\right] \tag{3.5}$$

Where $\alpha_j(x), \; and \;\; \beta_j(x)$ are assumed polynomial of the form

$$\alpha_j(x) = \sum_{i=0}^{i+m-1} \alpha_{j+1} x^i, \qquad \beta_j(x) = \sum_{i=0}^{i+m-1} \beta_{j+i} x^i \tag{3.6}$$

j ∈ {0, 1, ……, t-1 } j ∈ {0, 1, ….. , m-1}

where t >0 are arbitrarily chosen interpolation points taken from

$[x_n, x_{n+1}]$ and the collocation points $\bar{x}_j, \; j = 0, 1, \dots\dots m-1$ also belong to $[x_n \;\; x_{n+k}]$ Substituting the values of $\alpha_0(x), \; \alpha_1(x), \; \alpha_{\frac{3}{2}}(x), \; \beta_0, \; \beta_1, \; and \;\; \beta_2$ in (3.5) and simplified we obtain,

$$y(x) = \left[\frac{48((x-x_{n+1})^5 - 160h(x-x_{n+1})^3 + 37h^4(x-x_{n+1}))}{75h^2}\right] y_n$$

$$+ \left[\frac{-48\left((x-x_{n+1})^5 + 160h^2(x-x_{n+1})^3 + 87h^4(x-x_{n+1}) + 25h^5\right)}{25h^2}\right] y_{n+1}$$

$$+ \left[\frac{96\left((x=x_{n+1})^5 - 320h(x=x_{n+1})^3 + 224h^4(x-x_{n+1})\right)}{75h^5}\right] y_{n+\frac{3}{2}}$$

$$+\left[\frac{-42\left((x-x_{n+1})^5+25h(x-x_{n+1})^4+90h^2(x-x_{n+1})^3-23h^4(x-x_{n+1})\right)}{600h^3}\right]f_n$$
$$+\left[\frac{-252\left((x-x_{n+1})^5-50h(x-x_{n+1})^4+840h^2(x-x_{n+1})^3-300h^3(x-x^2-338h^4(x-x_{n+1})\right)}{600h^3}\right]f_{n+1}$$
$$+\left[\frac{6\left((x-x_{n+1})^5+25h(x-x_{n+1})^4+30h^2(x-x_{n+1})^3-11h^4(x-x_{n+1})\right)}{600h^3}\right]f_{n+2} \quad (3.7)$$

Evaluating (3.7) at $x = x_{n+2}$ the above continuous scheme reduces to

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{12}\{f_{n+2} + 10f_{n+1} + f_n\}. \quad (3.8)$$

This is the standard numerov method for the efficient solution of (2.1)

If we consider the first derivative function from (3.7) we have

$\frac{dy}{dx}(x) = Z(x), \quad \frac{dy}{dx}(a) = Z_0$ that is explicitly we obtain a scheme of the form,

$$hy'(x) + \frac{256}{75}y_{n+\frac{3}{2}} - \frac{153}{25}y_{n+1} + \frac{203}{75}y_n = \frac{h^2}{600}[9f_{n+2} + 522f_{n+1} - 63f_n] \quad (3.9)$$

Similarly if we evaluate the second derivative function of (3.7) at $x = x_{n+\frac{3}{2}}$ we obtained another discrete scheme of the form

$$\frac{48}{5}y_{n+\frac{3}{2}} - \frac{72}{5}y_{n+1} + \frac{24}{5}y_n = \frac{h^2}{30}\left[9f_{n+2} + 117f_{n+1} + 12f_n - 30f_{n+\frac{3}{2}}\right] \quad (3.10)$$

Thus the application of (3.8), (3.9) and (3.10) simultaneously provides the values of $y_1,\ y_2,\ and\ y_{\frac{3}{2}}$ at once without looking for any other method to provide y_1

ANALYSIS OF THE METHOD

To stat the initial value problem on the interval [x_0, x_2], we combine (3.8) when n=0 together with (3.9) and (3.10) explicitly we obtain the following set of equations

$$y_2 - 2y_1 + y_0 = \frac{h^2}{12}(f_2 + 10f_1 + f_0)$$

$$\frac{48}{5}y_{\frac{3}{2}} - \frac{72}{5}y_1 + \frac{24}{5}y_0 = \frac{h^2}{30}\left[9f_2 - 30f_{\frac{3}{2}} + 117f_1 + 12f_0\right] \quad (4.1)$$

$$\frac{256}{75}y_{\frac{3}{2}} - \frac{153}{25}y_1 + \frac{203}{75}y_0 = \frac{h^2}{200}[3f_2 + 174f_1 - 21f_0] - hZ_0$$

When (4.1) is put in matrix equation form, we have for n

$$\begin{bmatrix} -2 & 0 & 1 \\ -\frac{72}{5} & \frac{48}{5} & 0 \\ \frac{-153}{25} & \frac{256}{75} & 0 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{y+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -\frac{24}{5} \\ 0 & 0 & -\frac{203}{75} \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + h^2\left[\begin{bmatrix} \frac{10}{12} & 0 & \frac{1}{12} \\ \frac{117}{30} & -1 & \frac{9}{30} \\ \frac{174}{200} & 0 & \frac{3}{200} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{1}{12} \\ 0 & 0 & \frac{12}{30} \\ 0 & 0 & -\frac{21}{200} \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix}\right] \qquad 4.2$$

$A^{(0)}$ $A^{(1)}$ $B^{(0)}$ $B^{(1)}$

To normalize (4.2) for easy analysis, we multiply the matrices $A^{(0)}$, $A^{(1)}$ and $B^{(1)}$ by the inverse of $A^{(0)}$ i.e

$$\left[A^{(0)}\right]^{-1} = \begin{bmatrix} 0 & \frac{16}{45} & -1 \\ 0 & \frac{51}{80} & -\frac{3}{2} \\ 1 & \frac{32}{45} & -2 \end{bmatrix}$$

We obtain the normalized form as,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.\begin{bmatrix} y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.\begin{bmatrix} y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + h^2\left[\begin{bmatrix} \frac{31}{60} & -\frac{16}{45} & \frac{11}{120} \\ \frac{189}{160} & -\frac{51}{80} & \frac{27}{169} \\ \frac{28}{15} & -\frac{32}{45} & \frac{4}{15} \end{bmatrix}.\begin{bmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{89}{360} \\ 0 & 0 & \frac{33}{80} \\ 0 & 0 & \frac{26}{45} \end{bmatrix}.\begin{bmatrix} f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix}\right] \qquad (4.3)$$

Stability consideration for (4.3) is of the form,

$$\rho(\lambda) = \det\left[\lambda A^{(0)} - A^{(1)}\right]$$

$$\rho(\lambda) = \det\left[\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right] \qquad (4.4) \qquad \rho(\lambda) = \det\begin{bmatrix} \lambda & 0 & -1 \\ 0 & \lambda & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \rho(\lambda) = \lambda^2(\lambda - 1)$$

Which implies zero Stability. Since $|\lambda| \leq 1$

The block method (4.3) yield order p>1, hence is consistent.

Theorem (4.1) consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the method is consistent and zero stable, it implies the method is convergent for all point. C. baker, G.monegato, J. Pryce and G. V. Bergh (2001), Yusuph, Y. A. and Unumanyi P. (2002)

REMARK

For case K=2 with off grid at 3/2 interpolation point the summary of the block methods have the following order and error constant in tabular form.

Evaluating point	Order	Error Constant
X = x_{n+2} (3.8)	4	1/240
X = $x_{n+3\backslash 2}$ (3.9)	4	11/480
X = x_0 (3.10)	4	1/276

These schemes are zero - stable, Consistent and Convergent.

Conclusion/Recommendation

In this paper we discussed a block hybrid method of second derivative linear multistep method (LMM) which focusing in solving special second order ordinary differential equation. The analysis of the method was studied and found to be Stable, Consistent and Convergent. This study concluded that it has been shown in many literatures, the multistep hybrid method is very effective method for solving second order differential equation of the form (2.1).

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