Research Article

APPLICATION OF VARIATIONAL ITERATION METHOD FOR THE ANALYTICAL SOLUTIONS OF LINEAR AND NON LINEAR DELAY DIFFERENTIAL EQUATIONS USING LAGRANGE MULTIPLIER

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ABSTRACT

The paper employs the variational Iteration Method (VIM), to obtain analytical solution of Linear and Non linear Delay Differential equation (DDE). The method is base on the use of Lagrange multiplier for identification of optimal value of a parameter on a function, the procedure is a powerful tool for solving large amount of problems, using VIM it is possible to find the exact solution of the function. This technique provides a sequence of functions which converges to the exact solution of the problem. Convergence analysis is reliable enough to estimate the maximum absolute error of the approximate solution given by VIM. The method is an effective technique for choosing a reasonable initial solution designed in the solving process. Four examples are given to elucidate the method. Comparison is made between collocation splin method based on lagrange interpolation and the present VIM method with exact solution $u(t) = \lim_{n\to\infty} u_n(t)$, the solution is calculated in the form of a convergent series with an easy computable component. In conclusion the results show that VIM provide the straight forward and powerful mathematical tool for solving various linear and nonlinear differential equations of the nth order. The method is recommended due to it high accuracy and also can overcome the difficulties arising in the calculation of Adomian's polynomials, the method does not require small parameters which are needed in perturbation method.

Keywords: Variational Iteration Method, Analytical solutions, Linear and Non linear, Delay Differential Equation, Lagrange multiplier

INTRODUCTION

Many methods have recently been introduced to solve nonlinear problems, such as, VIM (T. A. Abessy, M. A. El-tawi, H. El-zoheiry. Adomial decomposition method (ADM) E. Fridman and E. Shustin (2011), D. J. Evans, K. R. Raslan (2013). Homotopy perturbation method M. Shadia (2012), N. H. Swailam, M. M. Khander (2010) and others M. M. Khadier (2011). The VIM is strongly and simply capable for solving a large class of linear differential equations without the tangible restriction of sensitivity to the degree of the nonlinear term and also it reduces the size of calculations besides, its interactions are direct and straightforward. A delay differential equation (DDE) is a differential equation in which the derivative of the function at any time depends on the solution at previous time. Introduction of the delay in the model enriches its dynamics and allows a precise description of the real life phenomena. DDEs are proved to be useful in control systems N. H. Sweilan, M. M. Khader, A. M. S. Mahdy (2012) lasers traffic models J. H. He, X. H. Wu (2013) metal cutting, epidemiology, neuroscience, population dynamics M. A. Ibrahim, A. El-Safty, S. M. Abo Hasha (2015), chemical kinetics M. A. Ibrahim, A. El- Safty, and S. M. Abo Hasha (2015) etc. in DDE one has to provide history of the system over the delay interval [τ, 0] as the initial condition. Due to this reason delay systems are infinite dimensional in nature. Because of infinite dimensionality the DDEs are difficult to analyze analytically M. Shadia (2012) and hence the analytical solutions play an important role. The main aim in this paper is to effectively employ VIM to establish exact solutions of DDEs and study the convergence of the method. To elucidate the method we present five examples of linear and non linear delay differential equations.

METHODOLOGY

We limit ourselves to consider the following nonlinear delay differential equation of the type

$$Lu(t) = f\left(t, u(t), u(\propto(t))\right), \quad 0 \le t \le T, \tag{1}$$

With the following initial conditions $u^k(0) = u^k \ k = 0,1,2\dots n-1, u(t) = \emptyset(t), t \le 0,$ (2)

Where the differential operator L is define by $L(.) = \frac{d^n(.)}{dt^n}$

Now to illustrate the analysis of VIM and study the analysis of convergence, we re-write Eq. (1) in the following form

$$Lu + Ru + N(u) = 0 \tag{3}$$

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With specified initial conditions, where L and R are linear bounded operators, ie, it is possible to find numbers $m_1, m_2 \succ 0$ where m_1 and m_2 are arbitrary constant such that $\|Lu\| \le m_1 \|u\|, \|Ru\| \le m_2 \|u\|.$ the nonlinear term N(u) is Lipchitz continuous with $|N(u) - N(v)| \le m|u - v|, \vee t \in j = [0, T],$ for arbitrary constant m > 0. the VIM give the possibility to write the solution of Eq. (3) with the aid of the correction function.

$$u_p = u_{p-1} + \int_0^t \lambda(\tau)[Lu_{p-1} + R\ddot{u}_{p-1} + N(\overline{u}_{p-1})]d\tau, \ p \ge 1 \tag{4}$$

It is obvious that the successive approximations $u_p \quad p \ge 0$ (the subscript p denotes the pth order approximation), can be established by determining λ, a general Lagrange multiplier λ that will be identified optimally via the variational theory. The function $\overline{u}_p$ is a restricted variation, which means $\delta \overline{u}_p = 0.$ therefore we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_p \quad p \ge 1$, of the solution u will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0, with λ determined, then several approximations $u_p \quad p \ge 1$, follow immediately. Consequently, the exact solution may be obtained by using

$$u = \lim_{p \to \infty} u_p. \tag{5}$$

In what follows, we apply VIM to some examples of linear and nonlinear delay differential equations to illustrate the strength of the method and to establish exact solutions exact solutions of these problems.

Now to illustrate how to find the value of the Lagrange multiplier λ, we will consider the following cases, which is dependent on the order of the operator L in Eq. (3), we will study the case of the operator L = $\frac{d}{dt}$ (without lost of generality), making the above correction functional stationary, and noticing that δ$u_p = 0$, we obtain $\delta u_p = \delta u_{p-1} + \delta \int_0^1 \lambda(\tau) \left[\frac{du_{p-1}}{d\tau} + R\overline{u}_{p-1} + N(\overline{u}_{p-1}) \right] \delta\tau$

$$= \delta u_{p-1} + [\lambda(\tau)\delta u_{p-1}]_{\tau=1} - \int_0^1 \lambda(\tau)[\delta u_{p-1}]d\tau = 0,$$

Where $\delta \overline{u}$ is considered as restricted variable ie $\delta \overline{u}_p = 0,$ yields the following stationary conditions

$$\lambda(\tau) = 0, \quad 1 + \lambda(\tau)_{\tau=t} = 0 \tag{6}$$

The first equation in (6) is called Lagrange- Euler equation and the second equation is called natural boundary condition. The solution of this equation gives the Lagrange multiplier $\lambda(\tau) = -1$. Now the following variational iteration formula can be obtained.

$$u_p = u_{p-1} - \int_0^1 [Lu_{p-1} + Ru_{p-1} + N(u_{p-1})]d\tau. \tag{7}$$

We start with an initial approximation, and by using the above iteration formula (7), we can obtain directly the other components of the solution.

CONVERGENCE ANALYSIS OF VARIATION ITERATION METHOD (VIM)

The sufficient conditions are presented to guarantee the convergence of VIM, when applied to solve nonlinear DDEs, where the main point is that we prove the convergence of the recurrence sequence, which is generated by using VIM.

Definition 1. The variation of a functional $v[u(x)]$ is defined as follows M. M. Khader (2011)

$$\delta v[u(x)] = \left[\frac{\partial}{\partial \alpha} v[u(x) + \alpha \delta u] \right]_{\alpha=0} \tag{8}$$

Where $v[u(x)]$ is a functional dependent on the function $u(x), \propto \in$ R

Theorem 1. If a functional $v[u(x)]$; has a variation,

achieves a maximum or a minimum at $u = u_0$, then at $u = u_0$ δv = 0, $\tag{9}$

where $u(x)$ is an interior point of the domain of definition of the functional.

Lemma 1. Let $A ; U \to V$ *be a bounded linar operator and let* $\{u_p\}$ be a convergent sequence in U with limit u, then $u_p \to u$ in U implies that $A(u_p) \to A(u)$ *in V.*

Proof. Since $\left\| Au_p - Au \right\|_v = \left\| A(u_p - u) \right\|_v \le \|A\| \left\| u_p - u \right\|_U,$

Hence $\lim_{p \to \infty} \left\| Au_p - Au \right\|_v \le \|A\| \lim_{P \to \infty} \left\| u_p - u \right\|_U = 0$ Implies that $A(u_p) \to A(u).$

Uniqueness theorem

Theorem 2. The nonlinear problem (3) has a unique solution, whenever $0 < \alpha < 1$, where $\alpha = (m_2 + m)T$ and the constants m2 and m are defined above.

Proof. Since, the solution of Eq. (3) can be written in the following form

$$u = f(t) - L^{-1}[Ru + N(u)],$$

Where $f(t)$ is the solution of the homogenous equation Lu = 0, and the inverse operator L^{-1} is define by $L^{-1}(.) = \int_0^t (.) dt$.

Now let, $u \ and \ u^*$ be two different solutions to 3 then by using the above equation, we get

$$\left| u - u^* \right| = \left| - \int_0^t [R(u - u^*) + N(u) - N(u^*)] dt \right|$$

$$\le \int_0^t \left[\left| R(u - u^*) \right| + \left| N(u) - N(u^*) \right| \right] dt \le (m_2 \left| u - u^* \right| + m \left| u - u^* \right|)T \le \alpha \left| u - u^* \right|$$

From which we get $(1 - \alpha) \left| u - u^* \right| \le 0$. since $0 < \alpha < 1$ then $\left| u - u^* \right| = 0$ implies, $u = u^*$

And this completes the proof.

Now, to prove the convergence of the VIM, we will rewrite Eq. 7 in the operator form as follows

$$u_p = A[u_{p-1}] \tag{10}$$

Where the operator A takes the following form

$$A[u] = - \int_0^t [Lu + Ru + N(u)] d\tau. \tag{11}$$

Convergence theorem

Theorem 3 (Banach's fixed point theorem). Assume that X be a Banach space and A; $X \to X$ is a nonlinear mapping and suppose that

$$\left\| A[u] - A[v] \right\| \le \gamma \left\| u - v \right\|, \qquad \forall u, v \in X, \tag{12}$$

For some constant $\gamma = (\alpha + m_1 T) < 1$. Then A has a unique fixed point. Furthermore, the sequence 10 ($u_p = A[u_{p-1}]$) using VIM with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A and

$$\left\| u_p - u_q \right\| \le \frac{\gamma^q}{1 - \gamma} \left\| u_1 - u_0 \right\|. \tag{13}$$

Proof. Denoting $(C[J], \| \ . \ \|)$ Banach space of all continuous functions on J with the norm defined by $\left\| f(t) \right\| = \max \left| f(t) \right|.$

We are going to prove that the sequence $\{u_p\}$ is a Cauchy sequence in this Banach space $\left\| u_p - u_q \right\| = \max \left| u_p - u_q \right|$

$$= \max \left| - \int_0^t [L(u_{p-1} - u_{q-1})] + R(u_{p-1} - u_{q-1}) + N(u_{p-1}) - N(u_{q-1})] d\tau \right.$$

$$\le \max \int_0^t [(m_1 + m_2 + m)(u_{p-1} - u_{q-1})] d\tau \le \gamma \left\| u_{p-1} - u_{q-1} \right\|.$$

Let $p = q + 1$ then

$$\|u_{q+1} - u_q\| \leq \gamma \|u_q - u_{q-1}\| \leq \gamma^2 \|u_{q-1} - u_{q-2}\| \leq \ldots\ldots \leq \gamma^q \|u_1 - u_0\|.$$

From the triangle inequality we have

$$\|u_p - u_q\| \leq \|u_{q+1} - u_q\| + \|u_{q+2} - u_{q+1}\| + \ldots\ldots + \|u_p - u_{p-1}\|$$

$$\leq \left[\gamma^q + \gamma^{q+1} + \ldots\ldots + \gamma^{p-1}\right] \|u_1 - u_0\|$$

$$\leq \gamma^q [\, 1 + \gamma + \gamma^2 + \ldots\ldots + \gamma^{p-q+1}] \|u_1 - u_0\| \leq \gamma^q \frac{1 - \gamma^{p-q-1}}{1 - \gamma}]\|u_1 - u_0\|.$$

Since $0 < \gamma < 1$ so, $(1 - \gamma^{p-q}) \angle 1$ then $\|u_p - u_q\| \leq \dfrac{\gamma^q}{1 - \gamma} \|u_1 - u_0\|.$

But $\|u_1 - u_0\| \angle \infty \quad so, \ as \ q \to \infty \ then \ \|u_p - u_q\| \to 0.$ we conclude that $\{u_p\}$ is a Cauchy sequence in C[J] so, the sequence is convergent. And this end the proof of the theorem.

Error estimate

Theorem 4. *The maximum absolute error of the approximate solution u_p to problem (3) is estimated to be*

$$\max|u_{exact} - u_p| \leq \beta, \tag{14}$$

Where $\beta = \dfrac{\gamma^q T[(m_1 + m_2)\|u_0\| + k]}{1 - \gamma}, \quad k = \max_{t \in J}|N(u_0)|.$

Proof. From theorem 3 and inequality 13 we have $\|u_p - u_q\| \leq \dfrac{\gamma^q}{1 - \gamma} \|u_1 - u_0\|,$

As $p \to \infty$ then $u_p \to u_{exact}$ and $\|u_1 - u_0\| = \max \left| - \int_0^t [Lu_0 + Ru_0 + N(u_0)] d\tau \right|$

$$\leq \max\int_0^t [|Lu_0| + |Ru_0| + |N(u_0)|] d\tau \leq T[(m_1 + m_2)\|u_0\| + k],$$

So, the maximum absolute error in the interval J is $\|u_{exact} - u_p\| = \max|u_{exact} - u_p| \leq \beta.$

This completes the proof.

NUMERICAL APPLICATION

To elucidate the method we discuss the numerical treatment of some problems of DDE (LDDE) and non linear DDE (NDDE) by using the propose method (VIM).

Example 1. Consider the LDDE of first order

$$\frac{du(x)}{dx} = \frac{1}{2} e^{\frac{x}{2}} u(\frac{x}{2}) + \frac{1}{2} u(x), \quad 0 \leq x \leq 1, \quad u(0) = 1 \tag{15}$$

The exact solution of this example is $u(x) = e^x$. To solve Eq.15 by means of VIM, we construct a correction functional which reads

$$u_{p+1}(x) = u_p(x) + \int_0^x \lambda(\tau) \left[u_{p\tau} - \frac{1}{2} e^{\frac{\tau}{2}} u_p(\frac{\tau}{2}) - \frac{1}{2} \bar{u}_p(\tau) \right] d\tau, \quad p \geq 0 \tag{16}$$

making the above correction functional stationary, and noticing that $\delta(0) = 0$ we obtain

$$\delta u_{p+1}(x) = \delta u_p(x) + \delta \int_0^x \lambda(\tau) \left[u_{p\tau} - \frac{1}{2} e^{\frac{\tau}{2}} u_p(\frac{\tau}{2}) - \frac{1}{2} u_p(\tau) \right] d\tau$$

$$= \delta u_p + [\lambda(\tau)\delta u_p]_{\tau=x} - \int_0^x \lambda(\tau)[\delta u_p] d\tau = 0$$

Where $\widetilde{\delta u}_p = 0$ is considered as a restricted variation, ie, $\widetilde{\delta u}_p = 0$, yields the following stationary conditions

$$\lambda(\tau) = 0, \quad 1 + \lambda(\tau)]_{\tau=x} = 0 \qquad (17)$$

The solution of this equation gives the Lagrange multiplier $\lambda(\tau) = -1$ now the following variational iteration formula can be obtained

$$u_{p+1}(x) = u_p(x) - \int_0^x \left[u_{p\tau} - \frac{1}{2}e^{\frac{\tau}{2}}u_p(\frac{\tau}{2}) - \frac{1}{2}u_p(\tau) \right] d\tau, \quad p \geq 0 \qquad (18)$$

We start with an initial approximation $u_0(x) = u(0)$, and by using the above iteration formula (18), we can obtain directly from the above components as

$$u_0(x) = 1, \quad u_1(x) = e^{\frac{x}{2}} + \frac{1}{2}x, \ ....$$

In order to verify numerically whether the proposed methodology leads to higher accuracy, we can evaluate the numerical solution using P = 5terms approximation. We achieved a very good approximation with the exact solution of Eq. (15) by using five terms only of the iteration equation derived above. The obtained numerical results justify the advantage of the proposed method, even in the few terms approximation is accurate.

Example 2. Consider the following initial value problem

$$y'(t) = -y(t) + 0.1y(0.8t) + 0.5y'(0.8t) + (0.32t - 0.5)e^{-0.8t} + e^{-t}, \quad t \geq 0,$$
$$y(0) = 0 \qquad (19)$$

To solve problem (19) by means of the VIM we have the following iteration formula

$$y_{n+1} = y_n(t) - \int_0^t e^{s-t}[y'_n(s) + y_n(s) - 0.1y_n(0.8s) - 0.5y'_n(0.8s) - (0.32s - 0.5)e^{-0.8s} - e^{-s}]ds. \qquad (20)$$

To get the iteration, we start with an initial approximation $y_0(t) = \varphi(t) = 0$ \qquad (21)

By means of the above iteration formula (20) we obtain

$$y_1(t) = 10.5e^{-t} + 1.6e^{-0.8t} - 10.5e^{-0.8t} + e^{-t}t, \quad y_2(t) = 7.064814815e^{-t} - 21e^{-0.8t} + e^{-t}t - 1.066666667e^{-0.64t} + 13.93518519e^{-0.64t},$$
$$y_3(t) = 4.793099982 \ e^{-t} - 14.12962963 \ e^{-0.8t} + e^{-t}t + 17.5e^{-0.64t} + 0.3846994537 \ e^{-0.512t}t - 8.163470348 \ e^{-0.512t}.$$

Note that $y_1(t), \ y_2(t), \ y_3(t)$ contain the common term $e^{-t}t, \ let \ y_0(t) = e^{-t}t,$

and substitute it into Eq. (19) we have $y_1(t) = e^{-t}t.$ \qquad (22)

It can be verified that $y(t) = e^{-t}t$ is indeed the exact solution of problem (19). As we show, we can obtain the exact solution of problem (19) by using only four iterations.

Remark (1) To improve efficiency, a simple technique is applied to choose more reasonable initial iteration functions in the solving process. Now, we introduce this technique as in example 19. We first choose the initial iteration function $y_0(t) = \varphi(t)$ as in 21, and solve problem 19 according to the following steps.

Step 1; By means of the iteration formula 20 and the initial iteration function $y_0(t)$, get the sequence $y_i(t), \ i = 1, 2, ..., N,$ where N is a given limited positive integer, for example

N = 3, 4, 5.

Step 2; Determine whether there exists a common maximal part $x(t) \ of \ y_i(t) \ (i = 1, 2, ..N)$ such that $y_i(t) = x(t) + \varphi(t),$
$i = 1, 2, ..., N.$ where $x(t)$ satisfies the initial condition of problem (19)
Step 3: If $x(t)$ exist verify whether

$$x(t) = x(t) - \int_0^t e^{s-t}[x'(s) + x(s) - 0.1x(0.8s) - 0.5x'(0.8s) - (0.32s - 0.5)e^{-0.8s} - e^{-s}]dt \qquad (23)$$

Is correct if Eq. (22) holds, then $x(t)$ is the solution of problem (19), if not, return to step 1, and choose $y_0(t) = x(t)$. if $x(t)$ doesn't exist, return to step 1, and choose $y_0(t) = y_N(t)$.

Example 3. Consider the NDDE of third-order.

$$\frac{d^3}{dx^3} u(x) = -1 + 2u^2(\frac{x}{2}), \quad u(0) = 0, \quad \frac{du(0)}{dx} = 1, \quad \frac{d^2u(0)}{dx^2} = 0, \quad 0 \le x \le 1 \tag{24}$$

To solve Eq. 23 by means of VIM, we can obtain the Lagrange multiplier $\lambda(\tau) = -\frac{1}{2}(\tau - x)^2$.

Now, the following variational iteration formula can be obtained

$$u_{p+1}(x) = u_p(x) - \int_0^x \frac{1}{2}(\tau - x)[u_{p\tau\tau\tau} - 2u_p^2(\frac{\tau}{2}) + 1]d\tau, \quad p \ge 0 \tag{25}$$

We start with an initial approximation $u_0(x) = x$, and using the above iteration formula (24), we can obtain the components of the solution $u(x)$. Consequently, the exact solution may be obtained by using

$$u = \lim_{p\to\infty} u_p = \sin(x). \tag{26}$$

We achieved a very good approximation with the actual solution of (23) by using 8 terms only of the iteration equation derived above.

Example 4. This example is concerned with the implementation of VIM to obtain the numerical solution of the Logistic equation with delay differential equation of the form

$$\frac{du(t)}{dt} = \rho u(t)(1 - u(t - r)), \quad t \angle 0, \quad \rho \angle 0 \tag{27}$$

With the following initial condition $u(0) = u_0, \quad u_0 > 0$

Applications of logistic equation. A typical application of the logistic equation is a common model of population growth. Let u(t) re- present the population size and t represent time where the constant ρ defines the growth rate. Another application of logistics equation is in medicine, where the logistic differential equation is used to model the growth of tumors. This application can be considered an extension of the above mentioned use in the framework of ecology. Denoting with u(t) the size of the tumor at time t.

To solve equation 26 by means of VIM we can obtain the Lagrange multiplier $\lambda(\tau) = -1$. Now the following variational iteration formula can be obtained as

$$u_{p+1}(t) = u_p(t) - \int_0^t [u_{p\tau}(\tau) - \rho u_p(\tau)(1 - u_p(\tau - r))]d\tau. \tag{28}$$

We start with an initial approximation $u_0 = 0.85$ and by using the above iteration formula 27

We can obtain the components of the solution u(t) with r = 0.0

$$u_0(t) = 0.85,$$
$$u_1(t) = 0.85 + 0.06375t,$$
$$u_2(t) = 0.85 + 0.06375t - 0.0111563t^2 - 0.000677344t^3$$

Therefore, the complete approximate solution can be readily obtained by the same iterative process. Consequently, the exact solution may be obtained by using,

$$u = \lim_{p\to\infty} u_p \tag{29}$$

In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the numerical solution using $p = 8$ terms approximation. The behavior of the approximate solution and the exact solution in the in the interval [0,6] at $\rho = 0.5$ using VIM prove high accuracy with r = 0.00, where in this case the exact solution is known and given by

$$u(t) = \frac{u_0}{u_0 + (1 - u_0)e^{-\rho t}}.$$

Also the approximate solution using different values of the parameter of delay r can be observed that the approximate solution depends on this parameter.

So far the approximate solution obtained justify the advantage of the proposed schemes even in the few terms approximation is accurate and the solutions are very rapidly convergent. It is evidence that the overall errors can be made smaller by adding new terms of the iteration formula. Also, it is noted that the VIM used here gives the possibility for obtaining analytical satisfactory solution for which the other techniques of calculation are more laborious and the results contain a great complexity.

Conclusion and Recommendation

The variational iteration method has been successfully applied to find the approximate solution of linear and nonlinear delay differential equations. The presented examples show that the result of the proposed method is in excellent agreement with those of adomian decomposition method. T. A. Abassy, M. A. El-tawi., H. El-Zoheiry,(2014) , but with less number of iteration. An interesting point about VIM is that only little iteration or, even in some special cases, one iteration, leads to exact solutions or solution with high accuracy. The main merits of VIM are, VIM can overcome the difficulties arising in the calculation of Adomian's polynomials in ADM, VIM does not require small parameters which are needed in perturbation method and no linearization is needed; hence the method is recommended because the method is very promising for solving wide application in nonlinear differential equations.

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