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IS THE CAUCHY DEFINITION OF LIMIT LIMITED IN RIGOR?A FOUNDATIONAL REVISIT

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ABSTRACT

The limit definition, or the ϵ-δ definition, as it has come down to us through two centuries, is still beset by suspicion from critics, being questioned for its level of rigor. The issue seems to stem from the precision of its statement and the logical soundness of its expression. There have been some questions on the soundness of its arguments in the midst of infinity and have called for a more 'discrete' approach. There have been arguments which say that the formal definition is invalid because 'infinity does not exist'. The aim of this paper is to show that, at least within the framework of mathematical analysis and the tenets of mathematical logic, the ϵ-δ definition is logically sound, and its level of precision has not been eroded by years of practice and advancement in the field, and in fact still serves as a spring board for further analytical studies.

Keywords: ϵ-δDefinition, Derivative, Darboux's theorem, Dini derivative, Denjoy-Young-Saks theorem, Caratheodory derivative.

INTRODUCTION

Lately I have been a witness to an online debate questioning the rigor of Cauchy's definition of the limit of a function at $x = x_0$One party to the debate was contending that the definition of the limit of a function as given by the French mathematician A.L. Cauchy (1789 – 1857) was inadequate not only in terms of establishing the value of the limit, but also in establishing the soundness of the proof as provided by the ϵ-δ definition. The 'debate' had become heated and as the exercise ensued the 'against' party started to get personal and began to throw language which was considered 'below the belt'. I will not name the parties involved, since I wanted to focus more on the subject of their debate, and on the merits of the opposite claims. The reader might as well be forewarned that our topic is not a new one, and my audience is definitely not the professional analysts who have been practicing their trade at the highest levels for the longest time. Such people not only have the highest appreciation of the details and ramifications of the concept of limits but are already quite comfortable in arriving at the deepest results brought about by the application of the formal definition. Rather, the hope is that by a sufficient examination of some of the consequences of the Cauchy definition, the curious student of analysis would be encouraged to further consider the subject after reading the article and therefore be a source of greater insight.

What was Cauchy thinking?

In his seminal work entitled Cours D'analyse (1821), Cauchy didn't give the actual$\epsilon - \delta$ definition of limit, but he hinted at something equivalent when he tried to define what a continuous function is:

> "…, the function f(x) is a continuous function of x between the assigned limits if, for each value of x between these limits, the numerical value of the difference$\mathrm{f}(\mathrm{x}+\alpha)-\mathrm{f}(\mathrm{x})$ decreases indefinitely with the numerical value of alpha. In other words, the function$\mathrm{f}(\mathrm{x})$ is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself." [1]

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This verbal formulation didn't exactly look like the $\epsilon - \delta$ definition that we know in modern times, but what is important is the fact that Cauchy was able to switch from verbal statements to expressions that involved mathematical inequalities when his proofs required him to do so [2]. Cauchy's appreciation of the elements was already deep, and reading through the proofs of some of his results clearly showed that he dispensed with the formal language of limits due to his command of the concept. From his perspective this amount of rigor was enough for him to be able to reach some of the deep and important results he was able to reach in his textbook.

We are not here to expound on the history behind the concept of the mathematical limit. What is important to us at this point is the fact that the origin of the rigor of limits reached its precise prototype in the work of Cauchy in Cours D'analyse, and every attempt after this was just an attempt to give it more mathematical formalism, the way we know it today. So, what is this modern formalism? It might be worthwhile to refresh our memory with the formal definition of the limit of a function. The function f(x) is said to have the limit L as x approaches a, symbolized by $\lim_{x\to a} f(x) = L$, if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$[3]. In symbolic shorthand, the whole statement can be represented as

$$\lim_{x\to a} f(x) = L \Leftrightarrow \forall\epsilon > 0\ \exists\delta > 0\ \ni \forall\ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

So,what is the issue?

When it comes to the limit of functions, modern pedagogy normally concentrates on applying the definition of the limit to particular examples where operations are normally "automatic". Automatic in the sense that, the candidate for the limit value of the function as x approaches some value$a$is often predictableand could be obtained by substituting the value $x = a$into the function. The next step is then to formally verify that it is indeed the limit as x approaches a by a direct use of the ϵ-δ definition. There is an argument, however, which says that the initial step of "guessing" the limit value in the preceding procedure involves the a priori presumption of a limit whose value is to be expected to satisfy the definition, and thus no longer becomes a cause for any surprise, making the entire argument circular .In other words, there is a perspective that poses the question as to how the

$\epsilon - \delta$ definition of the limit avoids circularity and maintains the independence of possible candidates for the limit value. This strikes at the heart of proof methodology in mathematical analysis, which relies on the principles of existence and uniqueness of objects, one of which is the limit. It is quite easy to use the formal definition to prove that *a value* is a limit of a function if one is confident enough that indeed such value is the limit. For example, it can easily be shown that the number $8\ (= 2^3)$ is the limit of the function $f(x) = x^3$ by using the formal definition:

To show:

$$\lim_{x\to 2} x^3 = 8 \Leftrightarrow \forall\epsilon > 0\ \exists\delta > 0\ \ni 0 < |x-2| < \delta \Rightarrow |x^3 - 8| < \epsilon$$

Now$|x^3 - 8| = |x-2||x^2 + 2x + 4|$;

Letting $|x-2| < 1$ for the moment, we proceed to find an upper bound for $|x^2 + 2x + 4|$.

(Note that, giving an upper bound of 1 for |x – 2| is valid, since the limit definition works best for small neighborhoods of x = 2. In fact, we can choose any upper bound between 0 and 1.)

$$|x^2 + 2x + 4| = |(x-2)^2 + 6(x-2) + 12|$$
$$\leq |x-2|^2 + 6|x-2| + 12$$
$$< 1 + 6 + 12 = 19.$$

From this we get $|x^3 - 8| = |x-2||x^2 + 2x + 4| < 19|x-2| < \epsilon$, with the rightmost inequality being the end result that we require. In other words, if we let $0 < |x-2| < \delta \equiv min\left\{1, \frac{\epsilon}{19}\right\}$, then we can retrace our steps to show that indeed, $|x^3 - 8| < \epsilon$ for any pre-assigned positive ϵ.Thus, we have shown that 8 is *a* limit for $f(x) = x^3$as x approaches 2, but true enough, this whole argument does not imply uniqueness of 8 as a limit. Is there a way of ruling out other candidates *by using the same $\epsilon - \delta$ definition*? This is one of the basic points which those who seek to replace the Cauchy definition bring to the fore as evidence that such formalism might be incomplete. The next question that has to be asked then is -- Is the Cauchy definition capable of ruling out unviable candidates as limits? Fortunately, the formal $\epsilon - \delta$ definition can still be used, as can be gleaned in the following illustration. For the sake of argument, let us take the limit of the function x^3 at $x = 2$to be equal to 7. Then

$$|f(x) - L| = |x^3 - 7| = |(x-2)^3 + 6(x-2)^2 + 12(x-2) + 1|.$$

From here we consider two cases:

Case 1.$x > 2$

If $x > 2$, then $|x^3 - 7| = |(x-2)^3 + 6(x-2)^2 + 12(x-2) + 1| = (x-2)^3 + 6(x-2)^2 + 12(x-2) + 1 > 1$.
Therefore, no amount of nearness of x to 1 will ever make $|x^3 - 7|$ close to 0.

Case 2. $x < 2$

If $x < 2$, then we can let $x = 2 - k,\ k > 0$, thus turning $|x^3 - 7| = |-k^3 + 6k^2 - 12k + 1| = |k^3 - 6k^2 + 12k - 1|$. It is enough to find values of k satisfying both
$k^3 - 6k^2 + 12k - 1 > 0$ and $k^3 - 6k^2 + 12k - (1+\epsilon) > 0$, of which we are certain do exist, in the light of Descartes' rule of signs, for any pre-assigned positive ϵ. We then get the inequality
$|x^3 - 7| = k^3 - 6k^2 + 12k - 1 > \epsilon$.

In other words, no amount of nearness of x to the number 2 will ever make $|x^3 - 7|$ close to 0. Consequently, a positive δ cannot be found that will satisfy the formal definition. Thus, we have shown that $lim_{x\to 2} x^3 \neq 7$. But you will notice that it took us a considerable amount of preparation just to rule out one number as a limit of the function. The procedure would be similar for the case of numbers not equal to 7, although admittedly it is impossible to rule out all the other numbers as limits. However, the $\epsilon - \delta$ definition itself in its generality gives us a way out of this dilemma, and thus enabling us to establish the uniqueness of the limit which we have obtained. Any standard course in mathematical analysis presents the proof of the uniqueness of the value, but we reproduce it here to show the level of rigor (and elegance) of the formal definition.

Proof of the uniqueness of the limit L.

Suppose there are two distinct limits L and L' for f(x) as x approaches a. Then each limit satisfies the $\epsilon - \delta$ definition.

$\Rightarrow \lim_{x\to a} f(x) = L \Leftrightarrow \forall\epsilon > 0\ \exists\delta_1 > 0\ \ni \forall\ 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{1}{2}\epsilon$.

Also,

$\Rightarrow \lim_{x\to a} f(x) = L \Leftrightarrow \forall\epsilon > 0\ \exists\delta_2 > 0\ \ni \forall\ 0 < |x-a| < \delta_2 \Rightarrow |f(x) - L'| < \frac{1}{2}\epsilon$.
$\Rightarrow |f(x) - L| + |f(x) - L'| = |f(x) - L| + |L' - f(x)| \leq |L' - L| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$,

Since $\epsilon > 0$ is arbitrary and $|L' - L|$ is a fixed positive quantity, the nature of ϵ enables us to choose a value much lower than $|L' - L|$ to arrive at a contradiction. (For example, let $\epsilon = \frac{1}{2}|L' - L|$.) Hence, when a particular value is found to satisfy the $\epsilon - \delta$ definition, we are certain that it is the one and only limitof the function for that point.

Proving a Number is Not a Limit: A Reprise

We want to demonstrate the fact that the Cauchy definition of limit can also be used to indirectly show that a number is not a limit of a function as x approaches a number a.For is purpose, we reproduce the formal definition in terms of symbols from mathematical logic:

$\lim_{x\to a} f(x) = L \Leftrightarrow \forall\epsilon > 0\ \exists\delta > 0\ \ni \forall\ 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

The negative (or negation) of this statement is:

$\lim_{x\to a} f(x) \neq L \Leftrightarrow \exists\epsilon > 0\ \forall\delta > 0\ \ni \exists\ 0 < |x-a| < \delta \Rightarrow |f(x) - L| \geq \epsilon$.

A direct translation of the preceding symbolism can be produced: A number L is not the limit of the function f(x) as x approaches $a$if and only if there exists a positive ϵ such that for any positive δ, there exists some x with $0 < |x-a| < \delta$ implying the inequality $|f(x) - L| \geq \epsilon$. To give an example, let us consider the common, but simple function

$$f(x) = \frac{|x|}{x} \Leftrightarrow f(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

and show that there is not any real number that can be the limit for this function as x approaches 0. In symbols, we wish to show that
$\lim_{x\to 0} f(x) \neq L \Leftrightarrow \exists\epsilon > 0\ \forall\delta > 0\ \ni \exists\ 0 < |x| < \delta \Rightarrow |\pm 1 - L| \geq \epsilon$.

The discussion can be broken down into several cases:

Case 1. $L > 1$.

If the limit is greater than 1, then it follows that$L - 1 = |L - 1| < L + 1 = |-1 - L|$. In other words, we can let $\epsilon = L - 1$, and for any neighborhood $0 < |x| < \delta$, one will always get $|f(x) - L| = |\pm 1 - L| \geq \epsilon$.

Case 2. $L < -1$.

If the limit is less than -1, then $-1 - L = |-1 - L| < 1 - L = |1 - L|$, hence we can let $\epsilon = -1 - L$, and for any neighborhood $0 < |x| < \delta$, one will always get $|f(x) - L| = |\pm 1 - L| \geq \epsilon$.

Case 3. $-1 < L < 1$.

If the limit is between -1 and 1, then $|1 - L| = 1 - L$and $|-1 - L| = L + 1$, and we can have $\epsilon = \min\{L + 1,\ 1 - L\}$, and again for any neighborhood $0 < |x| < \delta$, and again we get $|f(x) - L| = |\pm 1 - L| \geq \epsilon$.

Case 4. $L = 1$.

The limit cannot possibly be 1, since even though $|1 - L| = 0$ for any interval $(0, \delta)$, $|-1 - L| = 2$ for any interval $(-\delta, 0)$, and so we can choose ϵ to be any number from $(0,2]$. The case for $L = -1$ runs very similar, and so is omitted here.

The foregoing discussion concludes that the function $f(x) = \frac{|x|}{x}$ does not have a limit as x approaches 0, and demonstrates that the negation of the Cauchy definition for limits (which is a logically equivalent statement) can also be used to prove that the limit of a function does not exist at a particular value of x.

Is That the End of the Issue?

Apparently, the debate is not over, since the concept of limit naturally passes on to another equally important concept – the derivative of a function. The derivative can be treated as a special kind of limit, becoming the basis of much of mathematical analysis and having wide-ranging application, not only within mathematics, but also outside. It is no wonder that a lot of time and effort have been spent in making the foundations of the limit very precise and airtight, so that no questions would arise if and when it is applied to other fields of study. There have been attempts, however, to make the operations involving limits more similar to arithmetic, based only on familiar operations and requiring none of the infinite processes. There are still those who believe that this effort is still worth pursuing due to the insistence on the non-existence of infinite quantities. The issue becomes even more interesting and important on the pedagogic level since university students unwittingly commit the same error whenever they evaluate limits – there is a tendency of conveniently dropping the "$\lim_{x \to a}$" symbol and then outright substituting $h = 0$ into limit expressions when it is totally unwarranted. A probable reason behind this practice is expediency for some, and lack of appreciation for the concept for many. The concept of limit is very much the source of ambiguity in the evaluation of certain limits, giving rise to special quantities which are quite well known now as "indeterminate forms". The discussion about these quantities is not new, and one can see mistakes made by both professionals and enthusiasts in performing operations involving infinitesimal magnitudes. Take for example the indeterminate form 0^0. We cannot really pin down its "exact" value, because the 0 (either the base or exponent) could be interpreted as virtually or practically so small that its value is negligible. Aside from the fact that the two interpretations below can be made [4],

$$\lim_{x \to 0} x^0 = 1 \text{ and } \lim_{x \to 0^+} 0^x = 0,$$

0^0 can additionally be interpreted as something of "negligible^negligible", which only adds to the confusion. Thus 0^0 can be represented as being one of the following values: $0.0000001^{0.00000001}$, or $0.0000001^{-0.00000001}$, or $(-0.00000001)^{0.00000001}$, or $(-0.00000001)^{-0.00000001}$, and no one will be questioning your interpretation without the stipulation of any initial assumptions. Pick your poison on this one, so to speak. The point of this matter is that the concept of the infinitely small (or the infinitely big) could be tricky and touchy that some people would outright do away with them completely to avoid the heavy inconvenience of further analysis.

Now back to the derivative. The derivative of a function $f(x)$, symbolized as $f'(x)$,is defined as:

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

as is well known by now. (The derivative given in this form is sometimes also called Fréchet derivative.) In the definition above, the approach to 0 by h could be from positive numbers or negative numbers, in which case we could speak of *right – hand* derivative or *left-hand* derivative, respectively. As a particular case, if the derivative exists then the two-sided derivatives must exist and should be equal. But in general, the approach to 0 could be quite arbitrary. Just like in the case of general limits, the issue seems to stem from the *a priori* assumption on the derivative function that is used in the Cauchy definition (in addition to the objection to some of the infinitesimal operations involved because of the categorical rejection of the concept of the 'infinitely' small). Take again the case for the function $f(x) = x^3$. Parties who claim that derivative operations can be reduced to algebraic formulas conclude that rules can be set so as to make the determination of the derivative automatic. Immediately, they would say that the derivative of x^3 is $3x^2$. True enough, the foregoing exercise can be finished on a mere visit to any standard derivative formulas for the elementary functions. Even the application of its definition is simple enough, if one cares to do the requisite steps:

$$\frac{d}{dx} x^3 = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{3hx^2 + 3h^2 + h^3}{h}$$
$$= \lim_{h \to 0} (3x^2 + 3hx + h^2) = 3x^2$$

Note that, during the entire limiting process, I never dropped the limit prefix symbol, only omitting it during the step before the final evaluation, after making sure that the substitution of the $h = 0$ would give a valid expression (i.e., not indeterminate). The limit prefix is the license that enables one to divide by h during the entire limiting process, and should not be dropped at any point, since maintaining it pre-supposes that $h \neq 0$, and hence division by it is possible. As we have argued before regarding the answer obtained from a limiting process, the function that was obtained, technically, is at best *a* derivative of x^3. But as we have also shown in the preceding section, that limit value is the one and only that is possible, if it exists at all. This harkens back to the principles of existence and uniqueness of quantities or entities in mathematics which were mentioned earlier in this article. That being said, can the Cauchy definition be used to disprove that another function, say x^2, is *not* the derivative of x^3? Absolutely, as the two examples below will show:

Example 1.

Suppose the derivative of the function $f(x) = x^3$to be the function x^2.

$$\Rightarrow \left| \frac{(x+h)^3 - x^3}{h} - x^2 \right| = |2x^2 + 3hx + h^2|$$
$$= |x + h||2x + h|$$

$$= |x-(-h)||2x-(-h)| \geq \big||x|-|h|\big|\big|2|x|-|h|\big|$$
$$\geq (|x|-|h|)(2|x|-|h|)$$

Now if we consider all h values such that $0 < |h| < \frac{|x|}{2}$, then one obtains

$$(|x|-|h|)(2|x|-|h|) \geq \frac{|x|}{2} \cdot \frac{3|x|}{2} = \frac{3}{4}x^2$$

That is,
$\left|\frac{(x+h)^3-x^3}{h} - x^2\right| \geq \frac{3}{4}x^2.$
In other words, no matter how small h gets, the absolute value of the difference between the difference quotient and x^2 will be greater than an amount equal to $\frac{3}{4}x^2$. Consequently, no $\delta > 0$ will ever satisfy the Cauchy limit definition, and if we're going to look back at the negation of the Cauchy definition, we conclude that x^2 cannot be the derivative of x^3.

Example 2.
Suppose the derivative of $f(x) = e^x$ is the function x.
I start off the proof with the claim that $\frac{e^x-1}{x} > 1$ for all $x > 0$.
One will realize that this is the case after observing that $\frac{e^x-1}{x}$ is just the slope of the secant line from the point (0,1) to the point (x, e^x) on the graph of the function $f(x) = e^x$. Noting that the slope of the tangent line to e^x is just equal to $\frac{d}{dx}e^x\Big|_{x=0} = \lim_{h\to 0}\frac{e^h-1}{h} = 1$, as well as the fact that the secant lines emanating from the point (0, 1) to the point (x, e^x) are steeper than the tangent line at (0, 1) (hence have slopes greater than 1), it then follows that $\frac{e^x-1}{x} > 1$ for all $x > 0$.
Now on to the proof that x cannot possibly be the derivative of e^x. The negation of the Cauchy definition will again be used. For our purpose it will be sufficient to focus only on the interval $(x, x+\delta)$ for all $\delta > h > 0$.
$\Rightarrow \left|\frac{e^{x+h}-e^x}{h} - x\right| = \left|e^x\frac{e^h-1}{h} - x\right| > |e^x - x| = e^x - x.$

The last expression is devoid of h, and thus for every $\delta > 0$ such that $(x, x+\delta)$, letting $\epsilon = e^x - x$ then gives us $\left|\frac{e^{x+h}-e^x}{h} - x\right| > \epsilon$, and this finishes the proof.

Note that the two preceding examples give us a general procedure as to how we can prove that an *individual* function cannot be a derivative of a given function. Admittedly, this process is entirely unnecessary, as the definition of the Cauchy limit ensures us that once the *existence* of one candidate function is established, the *uniqueness* of the limit proves that it also the only derivative possible. The calculations were shown here to highlight the level of rigor and accuracy that comes in using the Cauchy definition, be it in the positive or the negative sense.

The Three D's (Darboux, Dini, and Denjoy et al)

The following discussion relates something to a property of the derivative, and doesn't deal exactly with foundational issues with limits. It is nonetheless part of an ongoing debate, and tells something about the level of understanding (or misunderstanding) regarding the nature of the derivative. The thing I refer to is the Mean Value Theorem for derivatives. Let us first have the formal statement of this important result:

Let $f(x)$ be continuous over the closed interval $[a, b]$ and differentiable over the open interval(a, b). Then$f'(c) = \frac{f(b)-f(a)}{b-a}$ for some c such that $c \in (a, b)$

Geometrically, this means that the secant line passing through the endpoints of the graph of the function over the interval [a, b] will be parallel to at least one tangent line to the graph of the function for some interior point. The mean value theorem (MVT) gives *sufficient* (and not necessary) conditions on the function $f(x)$for the consequence to hold true. Many students take the case of committing the mistake of giving conditions running against the theorem but arriving at the same consequence as evidence that the theorem is not true in general. There even are students who would wish to give a **restatement** of the Mean Value Theorem for derivatives, saying that a more precise expression of the result would be the following:

There exists $c \in (a, b)$ such that $f'(c) \cdot (b-a) = \int_a^b f'(x)dx$ for $f'(x)$ integrable over $[a, b]$.

There's nothing erroneous about the preceding statement in reality, as this is a direct statement involving the mean value theorem *for integrals*, with the derivative function $f'(x)$replacing the usual $f(x)$ instead. If the function $f'(x)$ has an antiderivative, or if $f'(x)$ is Riemann integrable, then it is straightforward to retrieve $f(x)$, at least hypothetically, giving $\int_a^b f'(x)dx = f(b) - f(a)$, and this brings us back to the original statement of mean value theorem *for derivatives.* The *unfavorable* aspect to the suggested restatement is mainly the chronological untimeliness it causes on the pedagogical timeline. It would sound odd and out of context to abruptly mention something involving the integral in a result when all previous discussions which led to it talked about derivatives only. The timeline for all academic discussions that culminate to the MVT points to the economy of delivery that has pervaded mainstream mathematics pedagogy, and in our case what we are really saying is that the concept of the integral is *not* needed for a full treatment and appreciation of the MVT *for derivatives*. Considering otherwise will render the entire order of the Calculus syllabus out of sync, which could be a much bigger issue than what it originally intended to improve on. One of the points we wish reiterate here is that, not satisfying the conditions of the MVT does not preclude the possibility of its consequence being true. This fact can be seen from the few examples given below.

Example 1. The function $f(x) = (x-1)^{1/3} + 3$ is continuous over the interval [0,2], but *not* differentiable over (0,2) (not entirely, at least). The endpoints of the graph are the points (0, 2) and (2, 4), and the slope of the secant line passing through these points is 1. Incidentally, a bit of algebra using its derivative function $f'(x) = \frac{1}{3(x-1)^{2/3}}$ will yield two values of $c = 1 \pm \frac{\sqrt{3}}{9}$.

Example 2. The function $f(x) = \begin{cases}(x-1)^{1/3}+3, & x \in [0,1]\\ (x-1)^{1/3}+4, & x \in (1,2]\end{cases}$ is an example of a function that is *not* continuous over [0, 2] and not *differentiable* over (0, 2) but nevertheless gives the same result as the MVT. The *two* values of c that give the same slope of $5/2$ of the secant line through the endpoints are $1 \pm \frac{2}{27}\sqrt{2}$. Note that each of the two pieces of the function yield the same derivative function over each of its interior, hence there is essentially one derivative equation to consider and solve.

Example 3. The function

$$f(x) = \begin{cases} x, & [0,1) \\ x-1, & [1,2] \\ x & (2,3] \end{cases}$$

is an example of a function which is again *not* continuous over a closed interval and *not* differentiable over the interior and gives the same conclusion as the MVT. The slope of the secant line through the endpoints is clearly 1, but for all points except the endpoints and $x = 1$ and $x = 2$, the derivative is also equal to 1. Hence there are infinitely many c values that satisfy the conclusion of the MVT in this case. One issue which makes it difficult to understand the Mean Value Theorem is its striking similarity to the statement of **Darboux's theorem**, which basically says that if a function is differentiable over the interval $[a, b]$ then it satisfies the intermediate value property [5]. (We have to give allowance here for the endpoints because we are dealing with a closed interval rather than an open one. Differentiability at the left endpoint means the right-hand derivative exists; while at the right endpoint, the left-hand derivative exists.) Hence, if c and d are numbers such that $a < c < d < b$ and w is a number such that $f'(c) < w < f'(d)$ then there is some $z \in (c, d)$ such that $f'(z) = w$. The two theorems give two different conditions and conclusions. Some students would say that the case of the infinite derivative at the point $x = 1$ in Example 1 violates the MVT's conclusion when they unwittingly have Darboux's theorem in mind. And apparently the state of affairs in this example undermines the truth of the MVT.In fact, all the possible 'statuses' of the derivative at any in an interval have already been predicted by a result known as the **Denjoy-Young-Saks theorem** (DYS). We will elaborate on this important result a little later. To have an appreciationDYS, we have to mention what are called **Dini derivatives** (or Dini derivates), which are important in analyzing the differentiability of the function at a point. The four Dini derivatives are defined as follows [6]:

$$D^+f(x) = \underset{x \to {x_0}^+}{\limsup} \frac{f(x) - f(x_0)}{h}$$

$$D_+f(x) = \underset{x \to {x_0}^+}{\liminf} \frac{f(x) - f(x_0)}{h}$$

$$D^-f(x) = \underset{x \to {x_0}^-}{\limsup} \frac{f(x) - f(x_0)}{h}$$

$$D_-f(x) = \underset{x \to {x_0}^-}{\liminf} \frac{f(x) - f(x_0)}{h}$$

They were named after the Italian mathematician Ulisse Dini (1845 – 1918). The DYS theorem [7] says that if f is a real-valued function defined on an interval, then with the possible exception of a set of measure 0 on the interval, the Dini derivatives of f satisfy one of the following four conditions at each point:

- *f has a finite derivative*
- *$D^+f = D_-f$ are finite, $D^-f = \infty$, and $D_+f = -\infty$.*
- *$D^-f = D_+f$ are finite, $D^+f = \infty$, and $D_-f = -\infty$.*
- *$D^-f = D^+f = \infty$, and $D_-f = D_+f = -\infty$.*

The DYS theorem is such an all-encompassing result that we ask, as a means of verifying its accuracy, how do we classify the interior points of the three examples above with respect to it? In Example 1, all points in the open interval (0, 2), except for $x = 1$, fall under type (i), in which $f'(x)$ is finite. For the point $x = 1$ itself, $D^-f(x) = D^+f(x) = +\infty$, and $D_-f(x) = D_+f(x) = 1$. Apparently, the point $x = 1$ doesn't fall under any type, but this situation is fine, since the measure of this single point is 0, and clearly this is consistent with the consequence as predicted by DYS.

In Example 2, one still obtains $D^-f(x) = D^+f(x) = +\infty$, but $D_-f(x) = 1$ and $D_+f(x) = 2$. The conclusion is still the same as that for Example 1.

The reader is now invited to classify the interior points with respect to DYS for Example 3. (For the point $x = 1$ in example 3, $D^+f(1) = D_+f(1) = 1$, while $D^-f(1) = 0$ and $D_-f(1) = -\infty$. Do these answers violate DYS? What about for$x = 2$?)

CONCLUSION

Our preceding exposition clearly brings out the versatility and rigor of the Cauchy definition of the limit. First brought out to the world in 1821, Cauchy's definition of the limit has withstood the test of 200 years of analysis and scrutiny, regardless of geography. It has become canon in all mathematical analysis syllabi for all levels of pedagogy and investigation in both the elementary and advanced levels. Any attempt to replace the Cauchy definition would have to stand not only the test of time, but also the test of mathematical breadth and depth. Turning to a particular case, the Cauchy definition has been so far precise in defining the derivative of a function, and is equally clear in giving meaning to differentiability of a function at a point. Any attempt at a replacement would have to show that it could be more precise in expression but flexible enough (for lack of a better word) to be used in all proofs and applications. Perhaps the closest a replacement could have got which involves the differentiability of a function is Caratheodory's definition of the derivative. (Which was named after the Greek mathematician Constantin Caratheodory (1873 – 1950).)

Caratheodory defined differentiability as follows [8]:

The function f is **differentiable** at $x = a \in U$, where U is an open interval, if there exists a continuous function $\varphi_a(x)$at $x = a$such that$f(x) - f(a) = \varphi_a(x)(x - a)$.

The Caratheodory definition is simpler than the Cauchy limit definition, and apparently doesn't make use of the limit notation. It is important to emphasize that the Caratheodory differentiability is equivalent to the Cauchy differentiability. However, Caratheodory takes direct advantage of the continuity of a function at a point, the foundations of which is still based on the Cauchy limit of a function.

(This can be deduced when we observe that $\varphi_a(x)$is continuous at$x = a$ making $\varphi_a(a)$finite, and consequently$f(x) \to f(a)$as$x \to a$.From this it is conclusive $f'(a) = \varphi_a(a)$.)The Caratheodory definition is powerful enough to prove the Chain Rule, the Inverse Function Theorem, and the Critical Point Theorem among other important things [9], but has not been attractive enough to the larger mathematical community so as to replace the current pedagogical approach which is based on the Cauchy definition of limit. It seems that any attempt at a replacement eventually becomes relegated to the status of "Further Studies", open to students for treatment only after a clear appreciation of the standard approach has been ensured, presumably not only because of the latter's thoroughness but as well as its time-tested precision and rigor. Until a replacement candidate has proven that it can do better on the basis of these grounds, the Cauchy definition is here to stay. And for a very long time in the future.

REFERENCES

1. Bradley, R.E. and Sandifer, C.E. (2009). Cauchy's Cours d'analyse: An Annotated Translation. Springer Science + Business Media. DOI10.1007/978-1-4419-0549-9.
2. Grabiner, J.V. (1983). Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus. The American Mathematical Monthly. 90(3), pp. 185 – 194. DOI 10.2307/2975545.
3. Peterson, T.S. (1955). Analytic Geometry and Calculus. Harper & Bros., New York. P.51.
4. Indeterminate form. (2021, May 26). In Wikipedia. https://en.wikipedia.org/wiki/indeterminate_form
5. Darboux's theorem. (2020, July 24). In Wikipedia. https://en.wikipedia.org/wiki/Darboux%27s_theorem_(analysis)
6. Stover, C. "Dini Derivative." From MathWorld—A Wolfram Web Resource, created by Eric w. Weisstein. https://mathworld.wolfram.com/DiniDerivative.html
7. Weisstein, E.W. "Denjoy – Saks – Young Theorem." From MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/denjoy-Saks-YoungTheorem.html
8. Kuhn, S. (1991). The Derivative á La Caratheodory. The American Mathematical Monthly, 98(1) pp.41. DOI: 10.2307/2324035.
9. ________. (1991) The Derivative á La Caratheodory. The American Mathematical Monthly, 98(1) pp.41-43. DOI: 10.2307/2324035.

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