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Research Article

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CALCULATION OF CURVATURE ON SMOOTH LOGICALLY CARTESIAN SURFACE MESHES USING MATLAB

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ABSTRACT

The definition of a regular surface in (to which we have limited our discussion) contains many parts that confuse its physical meaning. A regular surface is simply a smooth two dimensional surface at that looks flat when enlarged; Regular surfaces are a special case of varieties in Euclidean space. Our main goal in this paper is to show how basic geometric concepts (such as curvature) can be understood from complementary mathematical and computational points of view. This dual perspective enriches the understanding on both sides and leads to the development of practical algorithms for working with real geometric data. The results show how the adjusted parameters of curvature equation can effect on the shape and implement that using matlab.

Keywords: Curvature, Matlab, Surface, Geometery.

INTRODUCTION

The geometry offers different measures of curvature, including Gaussian and mean curvature. The combination of these curvature values enables the local surface type to be categorized [Yuen *et al.,* 2000]. Differential geometry deals with much of the same problems as Euclidean geometry, namely how to measure lengths, angles, and areas, but it is done on a more general level using the tools of calculus and linear algebra. These notes particularly emphasize the notion of curvature and the information they provide about geometric objects. We will cover the intuitive and rigorous definitions of curvature and what properties of curves and surfaces can be determined from the curvature [Sharpe Richard, 2000]. Approximations to the surface traditional and to the Gaussian curvature of a swish surface are typically needed once the surface is outlined by a collection of distinct points instead of by a formula. this example could arise in reverse engineering when correct measurements are made from the surface of a solid object. a customary technique in numerical analysis for examination approximations is straight line analysis. it's used here to match the errors of many approximations to the surface normal and to the Gaussian curvature. Below, the term curvature is employed to mean Gaussian curvature [Meek and Desmond, 2000]. Pre-computation geometry includes only the simplest curves: straight lines, dashed lines, arcs, ellipses, hyperbolas, and parabolas. These are the curves that form the basis of Greek geometry. Archimedes) were known in antiquity, the general theory of plane curves began only after the invention of the Cartesian coordinates by Descartes 1 at the beginning of the 17th [Gray *et al.,* 2017]. Our main goal is to show how basic geometric concepts (such as curvature) can be understood from complementary mathematical and computational points of view. This dual perspective enriches the understanding on both sides and leads to the development of practical algorithms for working with real geometric data

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Basic Definitions

The definition of a regular surface in R^3 (to which we have limited our discussion) contains many parts that confuse its physical meaning. A regular surface is simply a smooth two-dimensional surface at R^3 that looks flat when enlarged; Regular surfaces are a special case of varieties in Euclidean space [Lee, 2016]

Curvature Displaying:

Methods for displaying the curvature include normal vectors, contour lines, and color. Normal vectors can indicate the curvature of the surface in a length proportional to the radius of curvature. The contour lines include the reference plane and a series of parallel spaced planes. Planes with the surface lead to planar curves on the surface. These curves can help determine surface properties, saddle points appear as passages, and maxima and minima appear as circles. The curvature value corresponds to one end of the color spectrum and the maximum curvature value corresponds to the other end of the spectrum with a linear distribution in the middle. Color change represents a percentage change in curvature, including a logarithmic color scale [Mohammed *et al.,* 2005].

Curves in R^3:

Definition A curve $\alpha(t)$ in R^3 is a C^∞ map from an open sub set of R into R^3 The curve assigns to each value of a parameter $t \in R$, a point $\left(x^1(t), x^2(t), x^3(t)\right)$ in R^3

$$U \in R \overset{\propto}{\rightarrow} \mathrm{R}^3 \tag{1}$$

$$t \rightarrow \alpha(t) = \left(x^1(t), x^2(t), x^3(t)\right) \tag{2}$$

One can imagine that the parameter t represents the time and the curve α represents the trajectory of a point-shaped particle in motion [Lugo, Gabriel, 1998]

The Geometry of Surfaces :

Now that we have an idea of what a surface is, how do we know its geometry? One of the most important techniques in mathematics, in all science, is linear approximation, by which we mean the following. We realize that the nonlinear or curved object in question is too complicated to study directly, so we approach it through something linear: a line, a plane, a Euclidean space, then we study the linear object and deduce results about that original curved object. Of course, that's exactly what we do when we attach the Frenet linear frame to a curve. This process is also useful on topics ranging from differential equations to algebraic topology [Friedman *et al.,* 2004].

Curvature:

What is the curvature of a curve? Well, a line isn't curved at all; its curvature must be zero. A circle with a small radius is "more curved" than a circle with a large radius. Circles and lines have a constant curvature. Curves that are not circles or lines have a curvature that varies from point to point. **Definitions (5.1)**. Let C be a regular smooth curve in plane or in space with arc length parametrization $r_{al}: [0, l] \to R^i$ $t(s) = r'_{al}$ is a unit tangent vector to the curve at the point p_s. The vector function $t: : [0, l] \to R^i$ is called the unit tangent vector field moving along the curve. Now we analyze the information hidden in the derived vector field $t' = r''_{al}$ along the curve C. An application generates:

COROLLARY (5.1): (1) At every point p_s of the curve, the derivative $t'(s)$ is perpendicular to $t(s) = t'(s).t(s) = 0$.

(2) For a plane curve C the vectors $t'(s)\ and\ \hat{t}(s)$ are parallel [Auslander and Robert, 2012].

Topological surfaces

We will mainly be interested in the smooth and regular surfaces defined in the next section; however, we will seldom use the following general definition. A connected subset Σ in Euclidean space R^3 is called a topological surface (more precisely an embedded surface without limitation) if every point of $p\epsilon\Sigma$ allows a neighborhood W in Σ that can be parameterized by an open subset in the Euclidean plane; that is, there is an ineffectively continuous mapping $U \to W$ from an open set $U \subset R^2$ such that its inverse $W \to U$ is also continuous [Auslander and Robert, 2012].

Definition (7.1) A topological surface patch is a set $S \subset R^3$ such that there exists a homeomorphism $X : U \to S$ from an open subset $U \subset R^2$ onto S , that is We call X a parametrisation of S [Montiel *et al.,* 2009].

Calculation of the Curvature:

Components of the Acceleration Vector the above definition of curvature uses the arc length parameterize of a given curve, but in general you only have a regular parameterize $r : [a, b] \to R^i$ by hand. the curvature at a certain point P is "hidden" in the acceleration vector at this point. In relation to the selected parameterize $r(t)$, the acceleration vector $a(t)$ is the derivative of the velocity vector $v(t)$:

$$a(t) = v'^{(t)} = r''(t) \qquad (3)$$

We use the expression

$$v(t) = r'(t) = v(t)t(t) \qquad (4)$$

from $(t) = r'(t) = s'(t)r'_{al}(s(t))$ to calculate the acceleration vector a(t) using both the product and the chain rule from Prop.

$$\begin{aligned} a(t) = v'(t) &= v'(t)t(t) + v(t)t'(t) \\ &= v'(t)t(t) + v^2(t)k(t)n(t) \end{aligned} \qquad (5)$$

with $n(t)$ as in (5). The second factor $v(t)$ in (a). since $r(t) = r_{al}(s(t))$ and $(t\ o\ s)'(t) = s'(t)t'(s(t)) = v(t)k(t)n(t)$. Before using (5) to compute the curvature of a given curve, let's look at the following attractive interpretation in mechanics: Equation (5) produces a decomposition of the acceleration vector $a(t)$ into a tangential component $a_t(t)$ and a normal component $a_t(t)$

$$a(t) = a_t(t) = a_n(t) = v'(t)t(t) + v^2(t)k(t)n(t) \qquad (6)$$

In particular, the magnitude of the tangential component is: $|a_t(t)| = v'(t)$ which is the scalar acceleration, i.e., the rate of change of the speed. The magnitude of the normal component is: $|a_n(t)| = v^2(t)k(t)$ Therefore, the force acting on a particle perpendicular to its orbit is proportional to the square of its speed and the curvature of the curve. every motorist; When driving through a tight bend, you have to brake drastically to avoid strong normal forces. Every engineer who plans roads or railways needs to know this very well [Raussen, 2008].

Calculating Curvature

The formulae for $K\ and\ H$ for general $v\ and\ w$ may be particularized to $x_u\ and\ x_v$ when a patch x is given for M. Notice that we are introducing the following traditional notation:

$$E = x_u.x_u, \quad F = x_u.x_v, \quad G = x_v.x_v \qquad (7)$$

$$l = S(x_u).x_u, \quad m = S(x_u).x_v = S(x_v).x_u \quad n = S(x_v).x_v \qquad (8)$$

Then, replacing the general $v, w\ by\ x_u, x_v$ we have the two curvature formulas

$$K = \frac{((x_u).x_u)(S(x_v).x_v) - (S(x_u).x_v)(S(x_v).x_u)}{(x_u.x_u)(x_v.x_v) - (x_u.x_v)(x_v.x_u)} \qquad (9)$$

$$= \frac{GI + En - 2Fm}{2(EG - F^2)}$$

Example 1 (Hyperboloid of Two Sheets Curvature).

Let M denote the hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

parametrized by

$$x(u, v) = (a\ sinh\,u\ cos\,v, b\ sinh\,u\ sin\,v, e\ cosh\,u).$$

Then

$$x_u = (a\ cosh\,u\ cos\,v, b\ cosh\,u\ sin\,v, e\ sinh\,u)$$
$$x_v = (-a\ sinh\,u\ sin\,v, b\ sinh\,u\ cos\,v, 0)$$

and

$$x_u \times x_v = (-bc\ \sinh^2 u\ \cos v, -ac\ \sinh^2 \sin v\ , ab \sinh u\ \cosh v)$$

Dividing by $|x_u \times x_v|$

$$U = \frac{x_u \times x_v}{W}$$

where

$$W = \sqrt{b^2c^2 \sinh^4 u\ \cos^2 v + a^2c^2 \sinh^4 \sin^2 u\ \cos^2 v + a^2b^2 \sinh^2 u\ \cosh^2 u}$$

We then have

$$E = a^2 \cosh^2 u \cos^2 v + b^2 \cosh^2 u \sin^2 v + c^2 \sinh^2 u$$

$$F = -a^2 sinh\, u\, cosh\, u\, sin\, v\, cos\, v + b^2\, sinh\, u\, cosh\, u\, sin\, v\, cos\, v$$

$$G = a^2 \sinh^2 u \sin^2 v + b^2 \sinh^2 u \cos^2 v$$

with

$$\begin{aligned} EG - F^2 &= b^2c^2 \sinh^4 u \cos^2 v + a^2c^2 \sinh^4 \sin^2 u \cos^2 v \\ &\quad + a^2b^2 \sinh^2 u \cosh^2 u \\ &= W^2 \end{aligned}$$

The following second partials then give $I, m\ and\ n$.

$$\begin{aligned} x_{uu} &= (a\, sinh\, u\, cos\, v, b\, sinh\, u\, sin\, v, c\, cosh\, u) \\ x_{uv} &= (-a\, cosh\, u\, sin\, v, b\, cosh\, u\, cos\, v, 0) \\ x_{uu} &= (-a\, sinh\, u\, cos\, v, -b\, sinh\, u\, sin\, v, 0) \\ I &= x_{uu}.U \end{aligned}$$

$$= \frac{-abc \sinh^3 u \cos^2 v - abc \sinh^3 u \sin^2 v + abc\, sinh\, u \cosh^2 u}{W}$$

$$= \frac{abc\, sinh\, u}{W}$$

using $1 + \sinh^2 u = \cosh^2 u$

$$m = x_{uv}.U$$

$$= \frac{abc \sinh^2 u \cosh u\, sin\, v\, cos\, v - abc \sinh^2 u\, cosh\, u\, sin\, v\, cos\, v}{W}$$

$$= 0$$

$$n = x_{vv}.U$$

$$= \frac{abc \sinh^3 u \cos^2 v + abc \sinh^3 u \sin^2 v}{W}$$

$$= \frac{abc \sinh^3 u}{W}$$

Hence, we obtain the Gauss curvature

$$K = \frac{In - m^2}{EG - F^2} = \frac{a^2b^2c^2 \sinh^4 u}{W^4}$$

which we may write as

$$K = \frac{1}{\left(\frac{W^2}{abc \sinh^2 u}\right)^2}$$

where

$$\frac{W^2}{abc \sinh^2 u} = \frac{bc}{a} \sinh^2 u \cos^2 v + \frac{ac}{b} \sinh^2 u \sin^2 v + \frac{ab}{c} \cosh^2 u$$

Now the coordinate function of the paramateraztion are $x = a\, sinh\, u\, cos\, v$, $y = b\, sinh\, u\, sin\, v$ and $z = c\, cosh\, u$, so the reader can check that the Gauss curvature may be written in terms of $x,\ y\ and\ z$ as

$$K = \frac{1}{a^2b^2c^2\left[\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right]^2} \qquad [11]$$

Matlab for Example 1 :

```
clc;
clear all;
[X,Y,Z] = meshgrid(-10:0.5:10,-10:0.5:10,-10:0.5:10);
a=1;
b=1;
c=1;
M = - X.^2/a^2 - Y.^2/b^2 + Z.^2/c^2;
p=patch(isosurface(X,Y,Z,M,1));
set(p,'FaceColor','black','EdgeColor','none');
daspect([1 1 1])
view(3);
camlight
```

Input:
Case 1: meshgrid(-10:0.5:10,-10:0.5:10,-10:0.5:10);

Result for Case 1 :

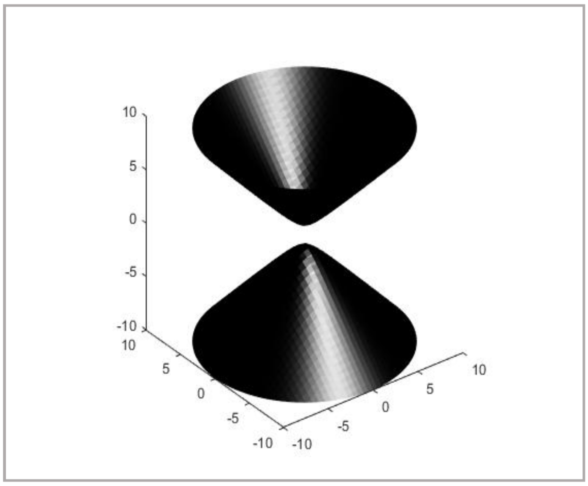

Fig. 1. Two sheet curvature

Input:
Case 2: meshgrid(-10:0.5:10,-10:0.5:10,0:0.5:10);

Result for Case 2 :

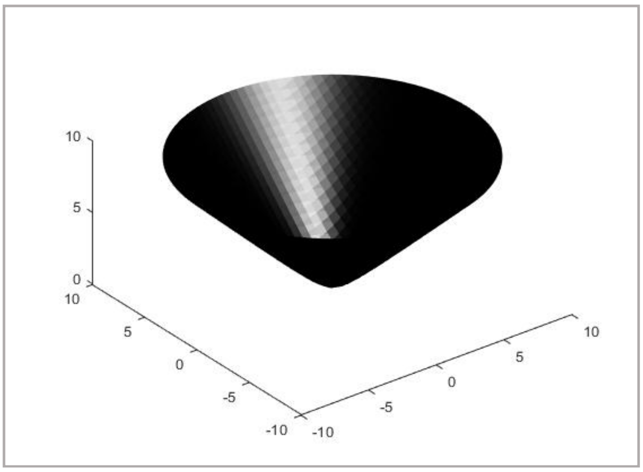

Fig. 2. One sheet curvature in positive way

Input:
Case 3: meshgrid(-10:0.5:10,-10:0.5:10,-10:0.5:0);

Result for Case 3

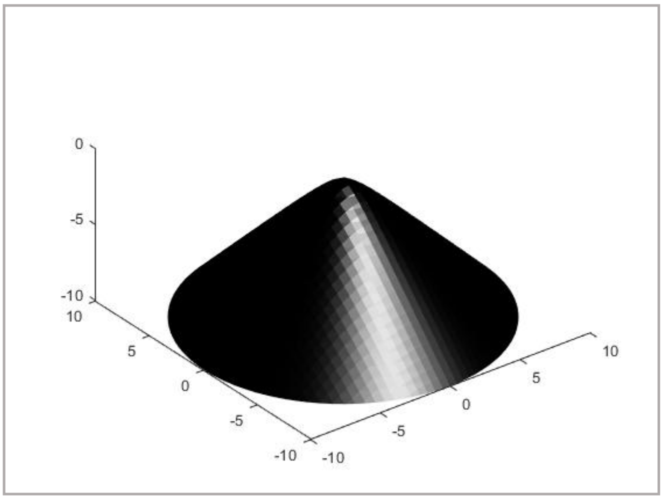

Fig. 3. One sheet curvature in negative way

CONCLUSION

In case 1 and case 2 and case 3 figures show the main results between two sheets curvature and one sheet curvature When we adjust the parameter of the equation

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

in 3 axis $[x, y, z]$ vectors as follows :

- Case1 $\mathrm{x}=[-10{:}\,0.5{:}\,10]$, $\mathrm{y}=[-10{:}\,0.5{:}\,10]$, $\mathrm{z}=[-10{:}\,0.5{:}\,10]$ in this case the equation(12) show this parameters as two faces curvature and figure(1) show the result (two sheet).
- Case 2 $\mathrm{x}=[-10{:}\,0.5{:}\,10]$, $\mathrm{y}=[-10{:}\,0.5{:}\,10]$, $\mathrm{z}=[0{:}\,0.5{:}\,10]$ in this case the equation(12) show this parameters in positive face curvature(upper curvature) or one sheet curvature as shown in figure(2)
- Case 3 $\mathrm{x}=[-10{:}\,0.5{:}\,10]$, $\mathrm{y}=[-10{:}\,0.5{:}\,10]$, $\mathrm{z}=[-10{:}\,0.5{:}\,0]$ in this case the equation(12) show this parameters in negative face curvature(lower curvature) or one sheet curvature as shown in figure(3)

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