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## Research Article

# EXISTENCE OF POSITIVE SOLUTION FOR A SIXTH–ORDER BEAM EQUATION WITH VARIABLE PARAMETERS

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**ABSTRACT**

This paper investigates the existence of positive solutions for the sixth-order boundary value problem with three variable parameters:

$$-u^{(6)} + A(t)u^{(4)} + B(t)u'' + C(t)u = u\varphi + f(t, u, u''),\ 0 < t < 1$$

$$-\varphi'' + \lambda\varphi = \mu g(t, u(t)),\ 0 < t < 1$$

$$u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = \varphi(0) = \varphi(1) = 0,$$

where μ is a positive parameter. The existence of the positive solution depends on μ, i.e. there exists a positive number $\bar{\mu}$ such that if $0 < \mu < \bar{\mu}$ the BVP has a positive solution. Using a fixed point theorem and an operator spectral theorem we give some new existence results.

***Keywords:*** Positive solutions; Variable parameters; Fixed point theorem; Operator spectral theorem.

## INTRODUCTION

Boundary-value problems for ordinary differential equations arise in different areas of applied mathematics and physics and the existence and multiplicity of positive solutions for such problems has become an important area of investigation in recent years; we refer the reader to [1-26] and the references therein. For example, the deformations of an elastic beam in the equilibrium state can be described as a boundary value problem of some fourth-order differential equations. Recently, boundary value problems for fourth-order ordinary differential equations have been extensively studied. It is well known that the deformation of the equilibrium state, an elastic beam with its two ends simply supported, can be described by the fourth-order boundary value problem:

$$u^{(4)}(t) = f(t, u(t), u''(t)),\ 0 < t < 1, i$$
$$u(0) = u(1) = u''(0) = u''(1) = 0. \qquad (1)$$

Existence of solutions for problem (1) was established for example by Gupta [15,16], Liu [17], Ma [18], Ma *et. al.,* [19], Ma and Wang [20], Aftabizadeh [21], Yang [22], Del Pino and Manasevich [23] (see also the references therein). All of those results are based on the Leray-Schauder continuation method, topological degree and the method of lower and upper solutions. In 2003, Li [24] studied the existence of positive solutions for the two-point boundary value problem with two constant parameters. Chai [25] established an existence result for the fourth-order boundary value problem with variable parameters. Recently, Wang and An [26] studied the existence of positive solutions for the second-order boundary value problem. It is well known that the deformation of the equilibrium state, an elastic circular ring segment with its two ends simply supported can be described by a boundary value problem for a sixth-order ordinary differential equation:

$$u^{(6)} + 2u^{(4)} + u'' = f(t, u),\ 0 < t < 1$$
$$u(0) = u(1) = u''(0) = u''(1) = u(4)(0) = u(4)(1) = 0,$$

However, there are only a handful of articles on this topic. In this paper we shall discuss the existence of positive solutions for the sixth-order boundary value problem

$$-u^{(6)} + A(t)u^{(4)} + B(t)u'' + C(t)u = u\varphi + f(t, u, u''),\ 0 < t < 1$$

$$-\varphi'' + \lambda\varphi = \mu g(t, u(t)),\ 0 < t < 1$$

$$u(0) = u(1) = u''(0) = u''(1) = u(4)(0) = u(4)(1) = 0,$$

$$\varphi(0) = \varphi(1) = 0, \qquad (2)$$

where $\lambda \geq -\pi^2$, A(t), B(t), C(t) ∈ C[0, 1] and μ is a positive parameter. Our results will generalize those established in [24,25,26]. More recently Li [27] studied the existence and multiplicity of positive solutions for the sixth-order boundary value problem with three variable coefficients. The main difference between our work and [27] is that we consider coupled system not only with three variable coefficients but also with a positive parameter μ. The existence of the positive solution depends on μ, i.e. there exists a positive number $\bar{\mu}$ such that if $0 < \mu < \bar{\mu}$ the BVP(2) has a positive solution. For this, we shall assume the following conditions throughout:

(H1) $f(t, u, v) : [0, 1] \times [0, \infty) \times (-\infty, 0] \to [0, \infty)$ and $g(t, u) : (0, 1) \times [0, \infty) \to [0, \infty)$ is continuous.

(H2) $a = \sup_{t\in[0;1]} A(t) > -\pi 2$, $b = \inf_{t\in[0;1]} B(t) > 0$, $c = \sup_{t\in[0;1]} C(t) < 0$, $\pi^6 + a\pi^4 - b\pi^2 + c > 0$,

where a, b, c ∈ R, $a = \lambda 1 + \lambda 2 + \lambda 3 > -\pi^2$, $b = -\lambda 1\lambda 2 - \lambda 2\lambda 3 - \lambda 1\lambda 3 > 0$, $c = \lambda 1\lambda 2\lambda 3 < 0$ and $\lambda 1 \geq 0 \geq \lambda 2 > -\pi^2$, $0 \leq \lambda 3 < -\lambda 2$.

Assumption (H2) involves a three-parameter no resonance condition.

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## PRELIMINARIES

Let $Y = C[0,1]$ and $Y_+ = \{u \in Y : u(t) \geq 0,\ t \in [0,1]\}$. It is well known that $Y$ is a Banach space equipped with the norm $\|u\|_0 = \sup_{t\in[0,1]} |u(t)|$.

We denote the norm $\|u\|_2$ by

$$\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}.$$

It is easy to show that $Z = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$ is complete with the norm $\|u\|_2$ and $\|u\|_2 \leq \|u\|_0 + \|u''\|_0 \leq 2\|u\|_2$.

Set $X = \{u \in C^4[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}$. For given $\chi \geq 0$ and $\nu \geq 0$, we denote the norm $\|\cdot\|_{\chi,\nu}$ by

$$\|\cdot\|_{\chi,\nu} = \sup_{t\in[0,1]} \left\{\left|u^{(4)}(t)\right| + \chi|u''(t)| + \nu|u(t)|\right\}, \quad u \in X.$$

We also need the space $X$ equipped with the norm

$$\|u\|_4 = \max\left\{\|u\|_0, \|u''\|_0, \left\|u^{(4)}\right\|_0\right\}.$$

In [11], it is shown that $X$ is complete with the norms $\|\cdot\|_{\chi,\nu}$ and $\|u\|_4$, and moreover $\forall u \in X$, $\|u\|_0 \leq \|u''\|_0 \leq \left\|u^{(4)}\right\|_0$.

**Lemma 1** ([28]). Let $E$ be a real Banach space and let $P$ be a closed convex cone in $E$. Let $\Omega$ be a bounded open set of $E$, $\theta \in \Omega$ and $Q : P \cap \overline{\Omega} \to P$ be completely continuous. Then the following conclusions are valid.

$(i)$ if $Qu \neq \nu u$ for every $u \in P \cap \partial\Omega$ and $\nu \geq 1$, then $i(Q, P\cap\Omega, P) = 1$,

$(ii)$ if mapping $Q$ satisfies the following two conditions

(a) $\inf_{u\in P\cap\partial\Omega} \|Qu\| > 0$

(b) $Qu \neq \nu u$ for every $u \in P \cap \partial\Omega$ and $0 < \nu \leq 1$,

then $i(Q, P\cap\Omega, P) = 0$.

**Lemma 2.** If $u(0) = u(1) = 0$ and $u \in C^2[0,1]$, then $\|u\|_0 \leq \|u''\|_0$, and so, $\|u\|_2 = \|u''\|_0$.

**Proof.** Since $u(0) = u(1)$, there is a $\alpha \in (0,1)$ such that $u'(\alpha) = 0$, and so $u'(t) = \int_\alpha^t u''(s)ds$, $t \in [0,1]$. Hence $|u'(t)| \leq \int_\alpha^t |u''(s)|\,ds \leq \int_0^1 |u''(s)|\,ds \leq \|u''\|_0$, $t \in [0,1]$. Thus $\|u'\|_0 \leq \|u''\|_0$. Since $u(0) = 0$, we have $u(t) = \int_0^t u'(s)ds$, $t \in [0,1]$, and so $|u(t)| \leq \int_0^1 |u'(s)|\,ds \leq \|u'\|_0$. Thus $\|u\|_0 \leq \|u'\|_0 \leq \|u''\|_0$. Since $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$ and $\|u\|_0 \leq \|u''\|_0$, we obtain that $\|u\|_2 = \|u''\|_0$.
This finishes the proof.

**Corollary 1.** $\forall u \in X$, $\|u\|_0 \leq \|u''\|_0 \leq \|u^{(4)}\|_0$, so we have $\|u\|_4 = \left\|u^{(4)}\right\|_0$.

**Corollary 2.** Let $r > 0$ and let $u \in \partial B_r \cap P$. Then $\|u\|_4 = \left\|u^{(4)}\right\|_0 = r$.

**Lemma 3.** [11] $(1+\chi+\nu)^{-1}\|\cdot\|_{\chi,\nu} \leq \|\cdot\|_4 \leq \|\cdot\|_{\chi,\nu}$, and $X$ is complete with respect to the norm $\|\cdot\|_{\chi,\nu}$, where the constants $\chi \geq 0$, $\nu \geq 0$.

For $h \in Y$, consider the following linear boundary value problem:

$$-u^{(6)} + au^{(4)} + bu'' + cu = h(t), \quad 0 < t < 1$$
$$u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0, \tag{3}$$

where $a, b, c$ satisfy the assumption

$$\pi^6 + a\pi^4 - b\pi^2 + c > 0 \tag{4}$$

and let $\Gamma = \pi^6 + a\pi^4 - b\pi^2 + c$. The inequality (4) follows immediately from the fact that $\Gamma = \pi^6 + a\pi^4 - b\pi^2 + c$ is the first eigenvalue of the problem $-u^{(6)} + au^{(4)} + bu'' + cu = \lambda u$, $u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0$ and $\phi_1(t) = \sin \pi t$ is the first eigenfunction, i.e. $\Gamma > 0$. Because the line $l_1 = \{(a, b, c) : \pi^6 + a\pi^4 - b\pi^2 + c = 0\}$ is the first eigenvalue line of the three-parameter boundary value problem $-u^{(6)} + au^{(4)} + bu'' + cu = 0$, $u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0$, if $(a, b, c)$ lies in $l_1$, then by the Fredholm alternative the existence of a solution of the boundary value problem (3) cannot be guaranteed.

Let $P(\lambda) = \lambda^2 + \beta\lambda - \alpha$ where $\beta < 2\pi^2, \alpha \geq 0$. It is easy to see that equation $P(\lambda) = 0$ has two real roots $\lambda_1, \lambda_2 = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}$, with $\lambda_1 \geq 0 \geq \lambda_2 > -\pi^2$. Let $\lambda_3$ be a number such that $0 \leq \lambda_3 < -\lambda_2$. In this case, (3) satisfies the following decomposition form:

$$-u^{(6)} + au^{(4)} + bu'' + cu = (-\frac{d^2}{dt^2} + \lambda_1)(-\frac{d^2}{dt^2} + \lambda_2)(-\frac{d^2}{dt^2} + \lambda_3)u, \quad 0 < t < 1. \tag{5}$$

It is obvious that $a = \lambda_1 + \lambda_2 + \lambda_3 > -\pi^2, b = -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3 > 0, c = \lambda_1\lambda_2\lambda_3 < 0.$

It is obvious that $a = \lambda_1 + \lambda_2 + \lambda_3 > -\pi^2, b = -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3 > 0, c = \lambda_1\lambda_2\lambda_3 < 0.$

Suppose that $G_i(t, s)(i = 1, 2, 3)$ is the Green's function associated with

$$-u'' + \lambda_i u = 0, \quad u(0) = u(1) = 0. \tag{6}$$

We need the following lemmas.

**Lemma 4** ([24]). Let $\omega_i = \sqrt{|\lambda_i|}$, then $G_i(t, s)(i = 1, 2, 3)$ can be expressed as

(i) when $\lambda_i > 0, G_i(t, s) = \left\{ \begin{array}{ll} \dfrac{\sinh \omega_i t \sinh \omega_i (1 - s)}{\omega_i \sinh \omega_i}, & 0 \leq t \leq s \leq 1 \\ \dfrac{\sinh \omega_i s \sinh \omega_i (1 - t)}{\omega_i \sinh \omega_i}, & 0 \leq s \leq t \leq 1 \end{array} \right\}$

(ii) when $\lambda_i = 0, G_i(t, s) = \left\{ \begin{array}{ll} t(1 - s), & 0 \leq t \leq s \leq 1 \\ s(1 - t), & 0 \leq s \leq t \leq 1 \end{array} \right\}$

(iii)when $-\pi^2 < \lambda_i < 0, G_i(t, s) = \left\{ \begin{array}{ll} \dfrac{\sin \omega_i t \sin \omega_i (1 - s)}{\omega_i \sin \omega_i}, & 0 \leq t \leq s \leq 1 \\ \dfrac{\sin \omega_i s \sin \omega_i (1 - t)}{\omega_i \sin \omega_i}, & 0 \leq s \leq t \leq 1 \end{array} \right\}.$

**Lemma 5** ([24]). $G_i(t, s)(i = 1, 2, 3)$ has the following properties:
(i) $G_i(t, s) > 0, \forall t, s \in (0, 1)$;
(ii) $G_i(t, s) \leq C_i G_i(s, s), \forall t, s \in [0, 1]$;
(iii) $G_i(t, s) \geq \delta_i G_i(t, t) G_i(s, s), \forall t, s \in [0, 1]$;
where $C_i = 1, \delta_i = \frac{\omega_i}{\sinh \omega_i}$, if $\lambda_i > 0; C_i = 1, \delta_i = 1$, if $\lambda_i = 0; C_i = \frac{1}{\sin \omega_i}, \delta_i = \omega_i \sin \omega_i$, if $-\pi^2 < \lambda_i < 0$.

In what follows, we shall let $D_i = \int_0^1 G_i(s,s)ds$.

Now, since

$$-u^{(6)} + au^{(4)} + bu'' + cu = (-\frac{d^2}{dt^2} + \lambda_1)(-\frac{d^2}{dt^2} + \lambda_2)(-\frac{d^2}{dt^2} + \lambda_3)u$$

$$= (-\frac{d^2}{dt^2} + \lambda_2)(-\frac{d^2}{dt^2} + \lambda_1)(-\frac{d^2}{dt^2} + \lambda_3)u = h(t), \quad (7)$$

the solution of boundary value problem (3) can be expressed by

$$u(t) = \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau)h(\tau)d\tau ds dv, \quad t \in [0,1]. \quad (8)$$

Thus, for every given $h \in Y$, the boundary value problem (3) has a unique solution $u \in C^6[0,1]$ which is given by (8).

We now define a mapping $T : C[0,1] \to C[0,1]$ by

$$(Th)(t) = \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau)h(\tau)d\tau ds dv, \quad t \in [0,1]. \quad (9)$$

Throughout this article we shall denote $Th = u$ the unique solution of the linear boundary value problem (3).

**Lemma 6.** $T : Y \to (X, \|\cdot\|_{\chi,\nu})$ is linear and completely continuous where $\chi = \lambda_1 + \lambda_3, \nu = \lambda_1\lambda_3$ and $\|T\| \le D_2$.

**Proof.** The proof of completely continuous is similar to the proof of Lemma 6 in [25], so we omit it. Next we will show that $\|T\| \le D_2$. Assume that $h \in Y$ and $u = Th$ is the solution the boundary value problem (3). It is clear that the operator $T$ maps $Y$ into $X$. Now for all $\forall h \in Y, u = Th \in X$, $u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0$. Using (7) it is easy to see that

$$-u'' + \lambda_i u = \int_0^1 \int_0^1 G_j(t,v)G_k(v,\tau)h(\tau)d\tau dv, \quad t \in [0,1]. \quad (10)$$

and

$$u^{(4)} - (\lambda_i + \lambda_j)u'' + \lambda_i\lambda_j u = \int_0^1 G_k(t,v)h(v)dv, \quad t \in [0,1]. \quad (11)$$

where $i,j,k = 1,2,3$ and $i \ne j \ne k$.

We will now show $\|Th\|_{\chi,\nu} \le D_2 \|h\|_0, \forall h \in Y,$ where $\chi = \lambda_1 + \lambda_3 \ge 0, \nu = \lambda_1\lambda_3 \ge 0$. For this, $\forall h \in Y_+$, let $u = Th$, and by Lemma 5, $u \in X \cap Y_+$. The equality (10) with the assumption $\lambda_2 \le 0$ implies that $u'' \le 0$. Similarly, the equality (11) with the assumptions $\lambda_2 + \lambda_3 < 0$ and $\lambda_2\lambda_3 \le 0$ implies that $u^{(4)} \ge 0$.

From (11) with $\chi = \lambda_1 + \lambda_3 \ge 0$, $\nu = \lambda_1\lambda_3 \ge 0$ and $u \ge 0$, $u'' \le 0$, $u^{(4)} \ge 0$ we immediately have

$$\left|u^{(4)}(t)\right| + \chi|u''(t)| + \nu|u(t)| = u^{(4)} - (\lambda_1 + \lambda_3)u'' + \lambda_1\lambda_3 u = \int_0^1 G_2(t,v)h(v)dv, \quad t \in [0,1]. \quad (12)$$

For any $h \in Y$, let $h = \widehat{h}_1 - \widehat{h}_2, u_1 = T\widehat{h}_1, u_2 = T\widehat{h}_2$, where $\widehat{h}_1, \widehat{h}_2$ are the positive part and negative part of $h$, respectively. Let $u = Th$, then $u = u_1 - u_2$. From the above, we have $u_i \ge 0, u_i'' \le 0, u_i^{(4)} \ge 0, i = 1,2$, and the following equality holds:

$$\left|u_i^{(4)}(t)\right| + (\lambda_1 + \lambda_3)|u_i''(t)| + \lambda_1\lambda_3|u_i(t)| = \int_0^1 G_2(t,v)h_i(v)dv = \widehat{H}\widehat{h}_i, \quad t \in [0,1], \quad i = 1,2. \quad (13)$$

So, from (13), we have

$$\left|u^{(4)}(t)\right| + (\lambda_1+\lambda_3)\left|u''(t)\right| + \lambda_1\lambda_3\left|u(t)\right| = \left|u_1^{(4)}(t) - u_2^{(4)}(t)\right|$$
$$+(\lambda_1+\lambda_3)\left|u_1''(t)-u_2''(t)\right| + \lambda_1\lambda_3\left|u_1(t)-u_2(t)\right|$$
$$\leq \left(\left|u_1^{(4)}(t)\right| + (\lambda_1+\lambda_3)\left|u_1''(t)\right| + \lambda_1\lambda_3\left|u_1(t)\right|\right)$$
$$+\left(\left|u_2^{(4)}(t)\right| + (\lambda_1+\lambda_3)\left|u_2''(t)\right| + \lambda_1\lambda_3\left|u_2(t)\right|\right)$$
$$= \widehat{H}\widehat{h}_1 + \widehat{H}\widehat{h}_2 = \widehat{H}\left|h\right| \leq D_2\left\|\left|h\right|\right\|_0 = D_2\left\|h\right\|_0 .$$

Thus $\|Th\|_{\chi,\nu} \leq D_2\|h\|_0$, and hence $\|T\| \leq D_2$. ∎

Suppose that $G(t,s)$ is the Green's function of the linear boundary value problem

$$-\varphi''(t) + \lambda\varphi(t) = 0, \quad \varphi(0) = \varphi(1) = 0. \tag{14}$$

Then, the boundary value problem

$$-\varphi''(t) + \lambda\varphi(t) = \mu g(t,u(t)), \quad \varphi(0) = \varphi(1) = 0,$$

can be solved by using Green's function, namely,

$$\varphi(t) = \mu\int_0^1 G(t,s)g(s,u(s))ds, \quad 0<t<1 \tag{15}$$

where $\lambda > -\pi^2$. Thus inserting (15) into the first equation of (2), we have

$$-u^{(6)} + A(t)u^{(4)} + B(t)u'' + C(t)u = \mu u(t)\int_0^1 G(t,s)g(s,u(s))ds + f(t,u(t),u''(t)),$$
$$u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0. \tag{16}$$

Throughout this paper, we assume additionaly that the continuous function $g(t,u) : (0,1)\times[0,+\infty) \longrightarrow [0,+\infty)$ satisfies

$(H_3)$

$$g(t,u) \leq g_1(t)g_2(u)$$

where $g_1 : (0,1) \to [0,+\infty)$ and $g_2 : [0,+\infty) \to [0,+\infty)$ is continuous.
Moreover,

$$0 < \int_0^1 G(s,s)g_1(s)ds < +\infty,$$

and for every $R>0$, there exists $\widehat{k}_2 > 0$ such that

$$g_2(x) \leq \widehat{k}_2 x, \quad 0 \leq x \leq R,$$

where $g_2(x) \neq 0$.

## MAIN RESULTS

**Theorem 1.** Assume that $(H_1), (H_2), (H_3)$ hold and $L = D_2K < 1$. If

$$\lim_{|v|\to 0+} \inf \min_{t\in[0,1]} \inf_{u\in[0,+\infty)} \left(f(t,u,v)/|v|\right) > \Gamma/\pi^2,$$

and

$$\lim_{|v|\to \infty} \sup \max_{t\in[0,1]} \sup_{u\in[0,+\infty)} \left(f(t,u,v)/|v|\right) < (1-L)\,\Gamma/\pi^2,$$

then there exists a positive number $\overline{\mu}$ such that if $0 < \mu < \overline{\mu}$ the boundary value problem (2) has a positive solution.

**Proof.** We consider the existence of a positive solution of (16) (the function $u \in C^6(0,1) \cap C^4[0,1]$ is a positive solution of (16), if $u \geq 0$, $t \in [0,1]$, and $u \neq 0$). It is easy to see that (16) is equivalent to the following boundary value problem:

$$-u^{(6)} + au^{(4)} + bu'' + cu = -\left(A(t)-a\right)u^{(4)} - \left(B(t)-b\right)u'' - \left(C(t)-c\right)u$$
$$+\mu u(t)\int_0^1 G(t,s)g(s,u(s))ds + f(t,u,u''). \tag{17}$$

For any $u \in X$, let

$$(Gu)(t) = -\left(A(t)-a\right)u^{(4)}(t) - \left(B(t)-b\right)u''(t) - \left(C(t)-c\right)u(t).$$

The operator $G : X \to Y$ is linear. By Lemmas 2 and 3, $\forall u \in X$, $t \in [0,1]$, we have

$$|(Gu)(t)| \leq \left[-A(t) - B(t) - C(t) - (-a-b-c)\right]\|u\|_4$$
$$\leq K\|u\|_4 \leq K\|u\|_{\chi,\nu}$$

where $K = \max_{t\in[0,1]}\left[-A(t) + B(t) - C(t) - (-a+b-c)\right]$, $\chi = \lambda_2 + \lambda_3 \geq 0$, $\nu = \lambda_2\lambda_3 \geq 0$. Hence $\|Gu\|_0 \leq K\|u\|_{\chi,\nu}$, and so $\|G\| \leq K$. Also $u \in C^4[0,1] \cap C^6(0,1)$ is a solution of (17) iff $u \in X$ satisfies $u = T(Gu + h_1)$, where $h_1(t) = \mu u(t)\int_0^1 G(t,s)g(s,u(s))ds + f(t,u,u'')$ i.e.

$$u \in X, \quad (I - TG)\ u = Th_1. \tag{18}$$

The operator $I-TG$ maps $X$ into $X$. From $\|T\| \leq D_2$ together with $\|G\| \leq K$ and condition $D_2K<1$, and applying the operator spectra theorem, we find that $(I-TG)^{-1}$ exists and bounded. Let $L=D_2K$.

Let $H=(I-TG)^{-1}T$. Then (18) is equivalent to $u=Hh_1$. By the Neumann expansion formula, $H$ can be expressed by

$$H=(I+TG+\ldots+(TG)^n+\ldots)\ T=T+(TG)T+\ldots+(TG)^n T+\ldots. \tag{19}$$

The complete continuity of $T$ with the continuity of $(I-TG)^{-1}$ guarantees that the operator $H:Y\to X$ is completely continuous.

Now $\forall h\in Y_+$, let $u=Th$, then $u\in X\cap Y_+$, and $u''\leq 0,\ u^{(4)}\geq 0$. Thus we have

$$(Gu)(t)=-\left(A(t)-a\right)u^{(4)}-\left(B(t)-b\right)u''-\left(C(t)-c\right)u\geq 0,\quad t\in[0,1].$$

Hence

$$\forall h\in Y_+,\quad (GTh)(t)\geq 0,\quad t\in[0,1] \tag{20}$$

and so $(TG)(Th)(t)=T(GTh)(t)\geq 0, t\in[0,1]$.

It is easy to see [25] that the following inequalities hold: $\forall h\in Y_+$,

$$\frac{1}{1-L}(Th)(t)\geq (Hh)(t)\geq (Th)(t)\ \ t\in[0,1], \tag{21}$$

moreover,

$$\|(Hh)\|_0\leq \frac{1}{1-L}\|(Th)\|_0. \tag{22}$$

For any $u\in Y_+$, let $Fu(t)=\mu u(t)\int_0^1 G(t,s)g(s,u(s))ds+f(t,u,u'')$. From $(H_1)$, we have that $F:Y_+\to Y_+$ is continuous. It is easy to see that $u\in C^4[0,1]\cap C^6(0,1)$ being a positive solution of (16) is equivalent to $u\in Y_+$ being a nonzero solution of

$$u=HFu. \tag{23}$$

Let $Q=HF$. Obviously, $Q:Y_+\to Y_+$ is completely continuous. We next show that the operator $Q$ has a nonzero fixed point in $Y_+$. Let

$$P=\left\{u\in X: u(t)\geq \delta_1(1-L)g_1(t)\|u\|_0,\ -u''(t)\geq \delta_1(1-L)g_1(t)\|u''\|_0,\ t\in[0,1]\right\},$$

where $g_1(t)=\frac{1}{C_1}G_1(t,t)$. It is easy to see that $P$ is a cone in $Y$. Now we show $QP\subset P$.

For $\forall u\in P$, let $h_1=Fu$, then $h_1\in Y_+$. From (21), $(Qu)(t)=(HFu)(t)\geq (TFu)(t),\ t\in[0,1]$. From Lemma 5 for all $u\in P$, we have

$$(TFu)(t)\leq C_1\int_0^1\int_0^1\int_0^1 G_1(v,v)G_2(v,s)G_3(s,\tau)(Fu)(\tau)d\tau ds dv,\quad \forall t\in[0,1].$$

Thus

$$\int_0^1\int_0^1\int_0^1 G_1(v,v)G_2(v,s)G_3(s,\tau)(Fu)(\tau)d\tau ds dv\geq \frac{1}{C_1}\|TFu\|_0. \tag{24}$$

Also from (22) and (24) we have

$$(TFu)(t)\geq \delta_1 G_1(t,t)\int_0^1\int_0^1\int_0^1 G_1(v,v)G_2(v,s)G_3(s,\tau)(Fu)(\tau)d\tau ds dv$$

$$\geq \delta_1 G_1(t,t)\frac{1}{C_1}\|TFu\|_0\geq \delta_1 G_1(t,t)\frac{1}{C_1}(1-L)\|Qu\|_0,\quad \forall t\in[0,1].$$

We have a similar type inequality for $(TFu)''(t)$. Hence $QP\subset P$.

From $\lim_{|v|\to 0+} \inf \min_{t\in[0,1]} \inf_{u\in[0,+\infty)} (f(t,u,v)/|v|) > \Gamma/\pi^2$, we can choose $\varepsilon > 0$ such that

$$\lim_{|v|\to 0+} \inf \min_{t\in[0,1]} \inf_{u\in[0,+\infty)} (f(t,u,v)/|v|) > \Gamma/\pi^2 + \varepsilon.$$

Then $\exists r > 0$ such that $f(t,x,y) > (\Gamma/\pi^2 + \varepsilon)|y|,\ t \in [0,1], 0 < |y| < r$. Let $\Omega_r = \{u \in P : \|u''\|_0 < r\}$. For any $u \in \partial\Omega_r$, we have $\|u''\|_0 = r,\ 0 < -u''(t) \le r, t \in [0,1]$, and so $f(t,u(t),u''(t)) > (\Gamma/\pi^2 + \varepsilon)(-u''(t)),\ t \in (0,1)$. By $-u''(t) \ge \delta \|u''\|_0 = \delta r, t \in [\frac{1}{4}, \frac{3}{4}]$, where $\delta = \delta_1(1-L)\min_{t\in[\frac{1}{4},\frac{3}{4}]} g_1(t)$, it follows that

$$f(t,u(t),u''(t)) > (\Gamma/\pi^2 + \varepsilon)(-u''(t)) \ge (\Gamma/\pi^2 + \varepsilon)\delta r, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Now we prove $\inf_{u\in\partial\Omega_r} \|(Qu)''\|_0 > 0$. For any $u \in \partial\Omega_r$, by (21) we have

$$\|(Qu)''\|_0 \ge \|Qu\|_0 \ge (Qu)\left(\frac{1}{2}\right) \ge (TFu)\left(\frac{1}{2}\right)$$

$$= \int_0^1\int_0^1\int_0^1 G_1(\frac{1}{2},v)G_2(v,z)G_3(z,\tau)\left[\mu u(\tau)\int_0^1 G(\tau,s)g(s,u(s))ds + f(\tau,u(\tau),u''(\tau))\right]d\tau dz dv$$

$$\ge (\Gamma/\pi^2 + \varepsilon)\delta r \int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2},v)G_2(v,z)G_3(z,\tau)d\tau dz dv \ge \frac{1}{8}(\Gamma/\pi^2 + \varepsilon)\delta r b_1 b_2 b_3 > 0, \qquad (25)$$

where $b_i = \min_{\frac{1}{4}\le t,s\le\frac{3}{4}} G_i(t,s)$. Therefore, $\inf_{u\in\partial\Omega_r} \|(Qu)''\|_0 > 0$.

Next we prove $\forall u \in \partial\Omega_r, 0 < \kappa \le 1, Qu \ne \kappa u$. Suppose the contrary, that $\exists u_0 \in \partial\Omega_r, 0 < \kappa_0 \le 1$, such that $Qu_0 = \kappa_0 u_0$. From (21) we get

$$u_0(t) \ge \kappa_0 u_0(t) = (Qu_0)(t) \ge (TFu_0)(t) = T\left(\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds + f(t,u_0(t),u_0''(t))\right)$$

$$= T\left(\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds\right) + T(f(t,u_0(t),u_0''(t))) \ge T(f(t,u_0(t),u_0''(t))), \quad t \in [0,1].$$

Let $v_0(t) = T(f(t,u_0(t),u_0''(t)))$. Then $u_0(t) \ge v_0(t)$ and $v_0(t)$ satisfies the BVP:

$$-v_0^{(6)} + av_0^{(4)} + bv_0'' + cv_0 = f(t,u_0(t),u_0''(t)), \quad 0 < t < 1. \qquad (26)$$

Multiplying (26) by $\sin(\pi t)$ and integrating over $[0,1]$ together with $v_0(0) = v_0(1) = v_0''(0) = v_0''(1) = v_0^{(4)}(0) = v_0^{(4)}(1) = 0$, $u_0(t) \ge v_0(t)$, we get

$$\Gamma\int_0^1 \sin(\pi s)u_0(s)ds \ge \Gamma\int_0^1 \sin(\pi s)v_0(s)ds = \int_0^1 \sin(\pi s)f(s,u_0(s),u_0''(s))ds; \qquad (27)$$

recall $\Gamma = \pi^6 + a\pi^4 - b\pi^2 + c$ is the first eigenvalue of the problem $-u^{(6)} + au^{(4)} + bu'' + cu = \lambda u$, $u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0$ and $\sin(\pi t)$ is an eigenfunction and note that we have equality in Eq. (27) since if we integrate by parts we have

$$\int_0^1 \sin(\pi s)f(s,u_0(s),u_0''(s))ds = \int_0^1 \sin(\pi s)\left(-v_0^{(6)}(s) + av_0^{(4)}(s) + bv_0''(s) + cv_0(s)\right)ds$$

$$= \int_0^1 (\pi^6 + a\pi^4 - b\pi^2 + c)\sin\pi s v_0(s)ds + \left[\sin\pi s\left(-v_0^{(5)}(s) + v_0^{(3)}(s)(\pi^2 + a) - v_0'(s)(\pi^4 + a\pi^2 - b)\right)\right]_0^1$$

$$+ \left[\cos(\pi s)\left(\pi v_0^{(4)}(s) + v_0''(s)(\pi^3 + a\pi) + v_0(s)(\pi^5 + a\pi^3 - b\pi)\right)\right]_0^1 = \Gamma\int_0^1 \sin(\pi s)v_0(s)ds.$$

From $f(t,u_0(t),u_0''(t)) > \left(\Gamma/\pi^2+\varepsilon\right)(-u_0''(t)),\ \ t\in(0,1)$, we have

$$\Gamma\int_0^1 \sin(\pi s)u_0(s)ds \geq (\Gamma/\pi^2+\varepsilon)\int_0^1 \sin(\pi s)(-u_0'')(s)ds = (\Gamma+\varepsilon\pi^2)\int_0^1 \sin(\pi s)u_0(s)ds. \tag{28}$$

Since $\int_0^1 \sin\pi s u_0(s)ds > 0$, we have $\Gamma \geq \Gamma+\varepsilon\pi^2$, a contradiction.

The above considerations together with Lemma 1 guarantee that $i\left(Q,\Omega_r,P\right)=0$.

From $\lim_{|v|\to+\infty}\sup\max_{t\in[0,1]}\sup_{u\in[0,+\infty)}\left(f(t,u,v)/|v|\right) < (1-L)\Gamma/\pi^2$, letting $N=(1-L)\Gamma$, we choose $0<\varepsilon<N/\pi^2$ such that $\lim_{|v|\to+\infty}\sup\max_{t\in[0,1]}\sup_{u\in[0,+\infty)}\left(f(t,u,v)/|v|\right) < \left(N/\pi^2-\varepsilon\right)$. Then $\exists R_0>0$, for $|y|\geq R_0$, $f(t,x,y) < \left(N/\pi^2-\varepsilon\right)|y|$, $t\in[0,1]$, $x\in[0,+\infty)$. Let us introduce the following notation: $M=\sup_{(t,x,|y|)\in[0,1]\times[0,R_0]\times[0,R_0]} f(t,x,y)$. Then

$$f(t,x,y) < \left(N/\pi^2-\varepsilon\right)|y| + M, \quad \forall t\in[0,1], \quad x,|y|\in[0,\infty).$$

Take $R>\max\{r,R_0,\frac{\sqrt{2}M}{\delta\varepsilon}\}$. Put $\Omega_R=\{u\in P: \|u''\|_0<R\}$. We prove $\forall u\in\partial\Omega_R$, $\nu\geq 1$, $\nu u\neq Qu$. Assume on the contrary that $\exists\nu_0\geq 1$, $u_0\in\partial\Omega_R$, $\nu_0u_0=Qu_0$. From (21) we have

$$u_0(t)\leq \nu_0u_0(t) = (Qu_0)(t) = (HFu_0)(t) \leq \frac{1}{1-L}(TFu_0)(t)$$

$$= \frac{1}{1-L}T\left(\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds + f(t,u_0(t),u_0''(t))\right)$$

$$= \frac{1}{1-L}T\left(\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds\right) + \frac{1}{1-L}T\left(f(t,u_0(t),u_0''(t)\right).$$

Let $v_0(t)=(TFu_0)(t)$. Then $u_0(t)\leq\frac{1}{1-L}v_0(t)$ satisfies the BVP:

$$-v_0^{(6)}+av_0^{(4)}+bv_0''+cv_0 = \mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds + f(t,u_0(t),u_0''(t)), \quad 0<t<1. \tag{29}$$

Multiplying (29) by $\sin(\pi t)$ and integrating over $[0,1]$ together with $v_0(0)=v_0(1)=v_0''(0)=v_0''(1)=v_0^{(4)}(0)=v_0^{(4)}(1)=0$, $u_0(t)\leq\frac{1}{1-L}v_0(t)$, we get

$$\Gamma\int_0^1 v_0(t)\sin(\pi t)dt = \int_0^1 \sin(\pi t)(\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds + f(t,u_0(t),u_0''(t)))dt$$

$$= \int_0^1 \sin(\pi t)\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))dsdt + \int_0^1 \sin(\pi t)f(t,u_0(t),u_0''(t))dt$$

$$\leq \mu\int_0^1 \sin(\pi t)u_0(t)\int_0^1 G(t,s)g(s,u_0(s))dsdt + \left(\frac{N}{\pi^2}-\varepsilon\right)\int_0^1(-u_0''(t))\sin(\pi t)dt + M\int_0^1\sin(\pi t)dt. \tag{30}$$

From (30), and using $G(t,s)\leq CG(s,s),\forall t,s\in[0,1]$ and $g(s,u_0(s))\leq g_1(s)g_2(u_0(s))\leq g_1(s)\widehat{k}_2u_0(s)$, we have

$$N\int_0^1 u_0(t)\sin(\pi t)dt \leq \mu\int_0^1 \sin(\pi t)u_0(t)\int_0^1 G(t,s)g_1(s)g_2(u_0(s))dsdt$$

$$+\left(\frac{N}{\pi^2}-\varepsilon\right)\int_0^1(-u_0''(t))\sin(\pi t)dt + M\int_0^1\sin(\pi t)dt$$

$$\leq \mu C\widehat{k}_2\int_0^1\sin(\pi t)\int_0^1 G(s,s)g_1(s)dsdt\,\|u_0\|_0^2 + M\int_0^1\sin(\pi t)dt$$

$$+N\int_0^1 u_0(t)\sin(\pi t)dt - \varepsilon\int_0^1(-u_0''(t))\sin(\pi t)dt.$$

Hence, using $\|u_0\|_0 \le \|u_0''\|_0$, we find

$$M \int_0^1 \sin(\pi t)dt + \mu C\widehat{k}_2 \int_0^1 \sin(\pi t)dt \int_0^1 G(s,s)g_1(s)ds \, \|u_0''\|_0^2$$

$$\ge \varepsilon \int_0^1 (-u_0''(t)) \sin(\pi t)dt \ge \delta\varepsilon \int_{\frac{1}{4}}^{\frac{3}{4}} \sin(\pi t)dt \, \|u_0''\|_0 ,$$

i.e.

$$M\frac{2}{\pi} + \mu C\frac{2}{\pi}\widehat{k}_2 k \, \|u_0''\|_0^2 \ge \delta\varepsilon \frac{\sqrt{2}}{\pi} \, \|u_0''\|_0$$

where $k = \int_0^1 G(s,s)g_1(s)ds > 0$. Thus $R = \|u_0''\|_0$, and we have

$$M + \mu C\widehat{k}_2 k R^2 \ge \frac{\delta\varepsilon}{\sqrt{2}} R,$$

hence

$$R^2 - \frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}}R + \frac{M}{\mu Ck\widehat{k}_2} \ge 0.$$

Solving

$$R^2 - \frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}}R + \frac{M}{\mu Ck\widehat{k}_2} = 0 \tag{31}$$

we find

$$R_1 = \frac{\frac{\delta\varepsilon}{\mu Ck\sqrt{2}} - \sqrt{\left(\frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}}\right)^2 - 4\frac{M}{\mu Ck\widehat{k}_2}}}{2}, \qquad R_2 = \frac{\frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}} + \sqrt{\left(\frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}}\right)^2 - 4\frac{M}{\mu Ck\widehat{k}_2}}}{2}$$

where $0 < R_1 < R_2$ if the discriminant $D = \left(\frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}}\right)^2 - 4\frac{M}{\mu Ck\widehat{k}_2} > 0$, i.e. $0 < \mu < \frac{\delta^2\varepsilon^2}{8Mk\widehat{k}_2 C}$. Let $\mu_1 = \frac{\delta^2\varepsilon^2}{8Mk\widehat{k}_2 C}$. Now, we can choose $\mu > 0$, such that $D > 0$, $R_0 < R_2$ and $r < R_2$ (it is always possible). For example, if we take

$$R_0 < \ R_2 = \frac{\frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}} + \sqrt{\left(\frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}}\right)^2 - 4\frac{M}{\mu Ck\widehat{k}_2}}}{2} \tag{32}$$

then we can rewrite (32) in the following form:

$$2R_0\mu < \frac{\delta\varepsilon}{Ck\widehat{k}_2\sqrt{2}} + \sqrt{\left(\frac{\delta\varepsilon}{Ck\widehat{k}_2\sqrt{2}}\right)^2 - 4\frac{\mu M}{Ck\widehat{k}_2}}. \tag{33}$$

It is easy to see that we can choose $\mu$ such that (32) is fulfilled, because if $\mu \to 0+$, then left side of (33) will tend to zero and the right side of (33) will tend to a finite positive number. Now because there exits a positive number $\mu_2$ such that (32) is fulfilled. Similarly we can see that $r < R_2$ is also fulfilled by choosing a suitable $\mu_3$.

We recall that $R > \max\{r, R_0, \frac{\sqrt{2}M}{\delta\varepsilon}\}$ and for fixed $R$, let us introduce the following notation:

$$\mu_4 = (\frac{\delta\varepsilon R}{\sqrt{2}} - M)\frac{1}{R^2Ck\widehat{k}_2}.$$

Since $R > \frac{\sqrt{2}M}{\delta\varepsilon}$, we have $M - \frac{\delta\varepsilon}{\sqrt{2}}R < 0$. Using (31), it is easy to see that if $0 < \mu < \mu_4$, then

$$R^2 - \frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}}R + \frac{M}{\mu Ck\widehat{k}_2} = R^2 + (M - \frac{\delta\varepsilon}{\sqrt{2}}R)\frac{1}{\mu Ck\widehat{k}_2} < 0$$

which is contradiction to

$$R^2 - \frac{\delta\varepsilon}{\mu Ck\widehat{k}_2\sqrt{2}}R + \frac{M}{\mu Ck\widehat{k}_2} \geq 0.$$

Let $\overline{\mu} = \min\{\mu_1, \mu_2, \mu_3, \mu_4\}$. From Lemma 1 we have $i(Q, \Omega_R, P) = 1$, and hence $i(Q, \Omega_R \backslash \overline{\Omega}_r, P) = 1$. Thus the boundary value problem (16) has a positive solution if $0 < \mu < \overline{\mu}$. ∎

**Theorem 2.** Assume that $(H_1), (H_2), (H_3)$ hold and $L = D_2K < 1$. If

$$\lim_{|v|\to+\infty} \inf \min_{t\in[0,1]} \inf_{u\in[0,+\infty)} (f(t,u,v)/|v|) > \Gamma/\pi^2,$$

and

$$\lim_{|v|\to 0+} \sup \max_{t\in[0,1]} \sup_{u\in[0,+\infty)} (f(t,u,v)/|v|) < (1-L)\Gamma/\pi^2,$$

then there exists a positive number $\overline{\mu}$ such that if $0 < \mu < \overline{\mu}$ the boundary value problem (2) has a positive solution.

**Proof.** From $\lim_{|v|\to 0+} \sup \max_{t\in[0,1]} \sup_{u\in[0,+\infty)} (f(t,u,v)/|v|) < (1-L)\Gamma/\pi^2$, letting $N = (1-L)\Gamma$, we can choose $0 < \varepsilon < N/\pi^2$ such that $\lim_{|v|\to 0+} \sup \max_{t\in[0,1]} \sup_{u\in[0,+\infty)} (f(t,u,v)/|v|) < N/\pi^2 - \varepsilon$. Thus $\exists r > 0$, $0 < |y| \leq r$, $x \in [0,+\infty)$, $0 \leq t \leq 1$ such that $f(t,x,y) \leq (N/\pi^2 - \varepsilon)|y|$. Put $\Omega_r = \{u \in P : \|u''\|_0 < r\}$. Now $\forall u \in \partial\Omega_r$, $f(t, u(t), u''(t)) < (N/\pi^2 - \varepsilon)(-u''(t))$, $t \in [0,1]$. We claim that $\forall u \in \partial\Omega_r$, $1 \leq \nu$, $Qu \neq \nu u$. Suppose the contrary, that $\exists u_0 \in \partial\Omega_r$, $1 \leq \nu_0$, $Qu_0 = \nu_0 u_0$. From (21), we have $(Qu_0)(t) \leq \frac{1}{1-L}(TFu_0)(t)$, $t \in [0,1]$. Letting $v_0 = TFu_0$, then

$$u_0(t) \leq \nu_0 u_0(t) = (Qu_0)(t) \leq \frac{1}{1-L}(TFu_0)(t) = \frac{1}{1-L}v_0(t) \tag{34}$$

and $v_0(t)$ satisfies the following BVP:

$$-v_0^{(6)} + av_0^{(4)} + bv_0'' + cv_0 = \mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds + f(t,u_0(t),u_0''(t)), \quad 0 < t < 1. \tag{35}$$

Multiplying (35) by $\sin(\pi t)$ and integrating on $[0,1]$ together with $v_0(0) = v_0(1) = v_0''(0) = v_0''(1) = v_0^{(4)}(0) = v_0^{(4)}(1) = 0$, $u_0(t) \leq \frac{1}{1-L}v_0(t)$, we get

$$\begin{aligned}\Gamma\int_0^1 \sin(\pi t)v_0(t)dt &= \int_0^1 \sin(\pi t)(\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds + f(t,u_0(t),u_0''(t)))dt \\ &= \int_0^1 \sin(\pi t)\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))dsdt + \int_0^1 \sin(\pi t)f(t,u_0(t),u_0''(t))dt\end{aligned}$$

and so using (34) and $G(t,s) \leq CG(s,s), \forall t,s \in [0,1]$, we have

$$N\int_0^1 u_0(t)\sin(\pi t)dt \leq \Gamma\int_0^1 v_0(t)\sin(\pi t)dt$$

$$\begin{aligned}&\leq \mu C\int_0^1 u_0(t)\sin(\pi t)dt\int_0^1 G(s,s)g_1(s)ds\widehat{k}_2\|u_0\|_0 + \left(\frac{N}{\pi^2} - \varepsilon\right)\int_0^1 (-u_0''(t))\sin(\pi t)dt \\ &= \mu C\int_0^1 u_0(t)\sin(\pi t)dtk\widehat{k}_2\|u_0''\|_0 + (N - \varepsilon\pi^2)\int_0^1 u_0(t)\sin(\pi t)dt.\end{aligned} \tag{36}$$

Since $\int_0^1 u_0(t)\sin\pi tdt > 0$, letting $k = \int_0^1 G(s,s)ds$, we have

$$N \leq \mu Crk\widehat{k}_2 + N - \varepsilon\pi^2. \tag{37}$$

Thus (37) is a contradiction if $\mu Crk\widehat{k}_2 < \varepsilon\pi^2$. In fact, then there is a positive number $0 < \overline{\mu} = \frac{\varepsilon\pi^2}{rCk\widehat{k}_2}$ so that if $0 < \mu < \overline{\mu}$ then (37) is a contradiction. Therefore $i(Q, \Omega_r, P) = 1$.

From $\lim_{|v|\to+\infty} \inf \min_{t\in[0,1]} \inf_{u\in[0,+\infty)} (f(t,u,v)/|v|) > \Gamma/\pi^2$, we choose $\varepsilon > 0$ with

$$\lim_{|v|\to+\infty} \inf \min_{t\in[0,1]} \inf_{u\in[0,+\infty)} (f(t,u,v)/|v|) > \Gamma/\pi^2 + \varepsilon.$$

Then $\exists R_0 > 0$ such that $f(t,x,y) > (\Gamma/\pi^2 + \varepsilon)|y|$ for $|y| \geq R_0$, $0 \leq t \leq 1$, and $x \in [0,+\infty)$. It is easy to see that $\exists M > 0$ such that $f(t,x,y) > (\Gamma/\pi^2 + \varepsilon)|y| - M$, for $t \in [0,1]$, $x, |y| \in [0,\infty)$. Take $R > \max\left\{r, \frac{R_0}{\delta}, \frac{\sqrt{2}M}{\varepsilon\delta}\right\}$ and put $\Omega_R = \{u \in P : \|u''\|_0 < R\}$. We show that $\inf_{u\in\partial\Omega_R} \|(Qu)''\|_0 > 0$, and $\forall u \in \partial\Omega_R$, $0 < \nu \leq 1$, $Qu \neq \nu u$.

For any $u \in \partial\Omega_R$, $-u''(t) \geq \delta\|u''\|_0 = \delta R > R_0$, $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$, and we have $f(t,u(t),u''(t)) \geq (\Gamma/\pi^2 + \varepsilon)(-u''(t)) \geq (\Gamma/\pi^2 + \varepsilon)\delta R$, $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Thus by (21) the following inequality holds:

$$\|(Qu)''\|_0 \geq \|Qu\|_0 \geq (Qu)\left(\frac{1}{2}\right) \geq (TFu)\left(\frac{1}{2}\right)$$

$$= \int_0^1\int_0^1\int_0^1 G_1(\frac{1}{2}, v)G_2(v,z)G_3(z,\tau)\left[\mu u(\tau)\int_0^1 G(\tau,s)g(s,u(s))ds + f(\tau,u(\tau),u''(\tau))\right]d\tau dz dv$$

$$\geq \int_0^1\int_0^1\int_0^1 G_1(\frac{1}{2}, v)G_2(v,z)G_3(z,\tau)f(\tau,u(\tau),u''(\tau))d\tau dz dv$$

$$\geq (\Gamma/\pi^2 + \varepsilon)\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2}, v)G_2(v,s)G_3(s,\tau)(-u''(\tau)d\tau ds dv \geq \frac{1}{8}(\Gamma/\pi^2 + \varepsilon)\delta R b_1 b_2 b_3 > 0, \tag{38}$$

where $b_i = \min_{\frac{1}{4}\leq t,s\leq\frac{3}{4}} G_i(t,s)$. Therefore, $\inf_{u\in\partial\Omega_R} \|(Qu)''\|_0 > 0$.

Suppose the contrary, $\exists u_0 \in \partial\Omega_R$, $0 < \nu_0 \leq 1$, such that $Qu_0 = \nu_0 u_0$. From (21) we find

$$u_0(t) \geq \nu_0 u_0(t) = (Qu_0)(t) \geq (TFu_0)(t) = T\left(\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds + f(t,u_0(t),u_0''(t))\right)$$

$$= T\left(\mu u_0(t)\int_0^1 G(t,s)g(s,u_0(s))ds\right) + T(f(t,u_0(t),u_0''(t)) \geq T(f(t,u_0(t),u_0''(t)), \quad t \in [0,1].$$

Let $v_0 = T(f(t,u_0(t),u''(t)))$. Then $u_0(t) \geq v_0(t)$ and $v_0(t)$ satisfies the boundary value problem:

$$-v_0^{(6)} + av_0^{(4)} + bv_0'' + cv_0 = f(t,u_0(t),u_0''(t)), \quad 0 < t < 1. \tag{39}$$

Multiplying (39) by $\sin(\pi t)$ and integrating over $[0,1]$ together with $v_0(0) = v_0(1) = v_0''(0) = v_0''(1) = v_0^{(4)}(0) = v_0^{(4)}(1) = 0$, $u_0(t) \geq v_0(t)$, we get

$$\Gamma\int_0^1 \sin(\pi s)u_0(s)ds \geq \Gamma\int_0^1 \sin(\pi s)v_0(s)ds = \int_0^1 \sin(\pi s)f(s,u_0(s),u''(s))ds. \tag{40}$$

From $f(t,u_0(t),u_0''(t)) > (\Gamma/\pi^2 + \varepsilon)(-u_0''(t)) - M$, $t \in (0,1)$, we have

$$\Gamma\int_0^1 \sin(\pi s)u_0(s)ds \geq \Gamma\int_0^1 \sin(\pi s)u_0(s)ds + \varepsilon\int_0^1 \sin(\pi s)(-u_0''(s))ds - M\int_0^1 \sin(\pi s)ds. \tag{41}$$

From (41) it follows that

$$M\int_0^1 \sin(\pi s)ds \geq \varepsilon \int_0^1 \sin(\pi s)(-u_0''(s)ds \geq \varepsilon\delta \left\|u_0''\right\|_0 \int_{\frac{1}{4}}^{\frac{3}{4}} \sin(\pi s)ds. \qquad (42)$$

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