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ON CLASSICAL SOLUTIONS OF A SINGULAR LINEAR DIFFERENTIAL EQUATION OF 2ND-ORDER IN THE SPACE K'.

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ABSTRACT

The main goal of this work is to describe all classical solutions of the homogeneous second order linear singular differential Euler equation in the space of generalized functions K'. We focus our investigations only to the situation called Euler case when it is realized respectively the condition m =r+2,n=r+1. For this aim, we look for the regular classical solutions by replacing the general form of the particular solution with the unknown λ, defined by y(x)=x^λ, into the investigated equation. Depending of the nature of the unknown λ we write the corresponding classical solution in the space K'. In this same work, we also use our previous results obtained in the case of the non homogeneous equation with Dirac delta function or it derivatives of s-order to write the general solution of the considered equation with dependence of various cases studied related to the relationships between the parameters.

Keywords: test functions, generalized functions, Dirac delta function, zero-centered solutions, classical solutions.

INTRODUCTION

Nowadays it is current that the modeling of several natural physical phenomena is carried out via differential equations and partial differential equations .We know that ordinary differential equations (ODE) are those in which the unknown function or functions depend on only one variable. There are several types which, depending of the category, can be solved easily and others with more difficulties. In the same way, when the unknown function contains several independent variables then, we, in this case speak of partial differential equations (PDE). Solving a differential equation is always the solution to undertake a new study once more and, then also a complete comprehension of a physical phenomena in our daily life. One may also understand and know that, it is not easy in some specific cases to solve certain kind of differential equations, even those of the first order. Solving differential equation is always conducting us in our mind to set a new kind of another differential equations to be investigated and solved. Depending of the spaces in which the solution of a differential equations are sought, we can face unwaited situations according to the form of the solutions. For exemple, a simple differential equation

$x^n y(x) = \delta^{(s)}(x)$ in the space *K'* admits a distributional solution defined by the formula

$$y(x) = \frac{\delta^{(s)}(x)}{x^n} = \frac{(-1)^n s!}{(s+n)!}\delta^{(s+n)}(x) + \sum_{k=1}^{n} C_k \delta^{(k-1)}(x)$$

where, C_k , $k = 1, \ldots\ldots n$ are arbitrary constants. For sufficient details see [18]. By this, we can see and note that, the solution of such equation is not unique and depends of $n$arbitrary constants.

When we used in our side Fourier transform to investigate for existence of zero-centered solutions of some kind of differential equations in our previous works, Amphon Liangprom and Kamsing Nonlaopon by their side applied Laplace transform to search the generalized solutions of the fourth order Euler differential equations $t^4 y^{(4)}(t) + t^3 y'''(t) + t^2 y''(t) + t y'(t) + m y(t) = 0$ where m is an integer and $t \in \mathbb{R}$. They got distributional solution and weak solution depending of the relationship between the parameter m and some constant $k \in \mathbb{N}$. For more details see [19]. Other researchers by their side as A. Kananthai , K. Nonlaopon, S. Suantai , V. Longani, H. Kim, A. M. Krall, R. P. Kanwal and L. L. Littlejohn also undertook got very interesting results on such topic related to various investigations of specific differential equations in the spaces of distributions. Details of such results can be found in [22,23, 24, 25, 26, 27]. Following our recent work, titled «On generalized-function solutions of a first order linear singular differential equation in the space K' via Fourier transform.» published in the Journal of Mathematical Sciences: Advances and Applications.Volume 70, 2022, Pages 27-51. Available at http://scientificadvances.co.in. DOI:http://dx.doi.org/10.18642/jmsaa_7100122244. in which, we completely investigated the question of the solvability of the singular linear differential equation of the first order $Ax^p y'(x) + Bx^q y(x) = \delta^{(s)}(x)$ and also, we dedicated an important place to classical solutions to it homogeneous equation, we arrived to analyze them step by step in the space K'. For details see also [18]. This upstair work already done can serve to us as a fundament to generalize such topic, when we turn to the second order linear differential equation $\mathrm{A}x^m y''(x) + Bx^n y'(x) + Cx^r y(x) = \delta^{(s)}(x)$ investigated on the question of the existence of zero-centered solutions in the space *K'* in the Euler case. For more details see our recent publication [18]. All what have been said naturally conducts us to set the problem of the existence and analysis of all classical solutions of the considered equation in the space *K'* . So that here in this work, we consider the following homogeneous linear singular differential equation of the second order defined by:

$$\mathrm{A}x^m y''(x) + Bx^n y'(x) + Cx^r y(x) = 0, \qquad (1).$$

where $A, B, C \in \mathbb{R}, m, n \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ y$is the unknown generalized function from the space K' and the parameters m, n and r are such that: $m = n + 1, n = r + 1$. In this case, the

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homogeneous equation takes the following form, to which we easily refeer sometimes within the work:

$$\mathrm{A}x^{r+2}\, y''(x) + Bx^{r+1}y'(x) + Cx^r\; y(x) = 0 \qquad (2).$$

This paper is structured as follow: Section 2 is devoted to well known concepts and notions related to some important facts of general theory of differential equation and generalized functions in the space K'. Presenting the general results of our investigation, we are given in section 3 the overview of the whole researches conducted with the formulated theorems and description of the general solution of the corresponding non homogeneous equation with Dirac delta function or it derivatives of $s-order$in the right hand side.We summarize and conclude our investigation in section 4 dedicated to the conclusion.

PRELIMINARIES

Before we proceed to our main results, the following definitions and concepts well known from the theory of generalized functions are required. For more details we can refeer and see also [2,3, 9,19,21].

Definition 2.1. Let K be the space consisting of all real-valued functions $\varphi(x)$ with continuous derivatives of all orders and compact support. The support of $\varphi(x)$ is the closure of the set of all elements$t \in \mathbb{R}$ $such\ that\quad \varphi(x) \neq 0. Then, \varphi(x)\ is\ called\ a\ test\ function.$

Definition 2.2. A distribution T is a continuous linear functional on the space K on the space of the real-valued functions with infinitely-differentiable and bounded support. The space of all such distributions is denoted by K'.

For every $\mathrm{T} \in K'$ *and* $\varphi(\mathrm{x}) \in K$,the value T has on $\varphi(\mathrm{x})$ is denoted by $(\mathrm{T}, \varphi(\mathrm{x}))$. Note that $(\mathrm{T}, \varphi(\mathrm{x})) \in \mathbb{R}$.

Below, let us give some examples of distributions.

(a) A locally-integrable function $\mathrm{g(x)}$ is a distribution generated by the locally-integrable function $\mathrm{g(x)}$. Here, we define

$$(\mathrm{g(x)}, \varphi(x)) = \int \mathrm{g(x)}\varphi(x)dx\ \ integration\ on\ the\ support\ \Omega, and\ \varphi(x) \in K\ .$$

In this case the distribution is called regular distribution.

(b) The Dirac delta function is a distribution defined by$(\delta(\mathrm{x}), \varphi(\mathrm{x})) = \varphi(0)$, and the support of $\delta(\mathrm{x})$ is $\{0\}$.

In this case the distribution is called irregular distribution or singular distribution.

Definition 2.3. The sth-order derivative of a distribution T, denoted by $T^{(s)}$, *is defined by* $\left(T^{(s)}, \varphi(x)\right) = (-1)^s \left(T, \varphi(x)^{(s)}\right)$ *for all*$\varphi(x) \in K$.

Let give an exemple of derivatives of the singular distribution

$\mathrm{T} = \delta$ we have:

a) $(\delta'(\mathrm{x}), \varphi(\mathrm{x})) = -(\delta(\mathrm{x}), \varphi'(\mathrm{x})) = -\varphi'(0)$;
b) $\left(\delta^{(s)}, \varphi(\mathrm{x})\right) = (-1)^s \left(\delta(\mathrm{x}), \varphi(\mathrm{x})^{(s)}\right) = (-1)^s\, \varphi(0)^{(s)}$.

Definition 2.4. Let $\omega(x)$ be an infinitely-differentiable function. We define the product of $\omega(x)$ with any distribution T in K' by $(\omega(x)T, \varphi(x)) = (T, \omega(x)\varphi(x))$*for all*$\varphi(x) \in K$.

Now we can move to the important part of the work stated in the following section.

MAIN RESULTS.

In this section, we present in full details the main results of our work based onto the research of classical solutions of the considered equation which takes the form (2). We look for classical (regular) solutions as follow: $y(x) = x^\lambda$, where λ −undetermined coefficient. Setting $y(x)$ and its derivatives into (2) leads us to the following result:

$$\mathrm{A}x^{r+2}\lambda(\lambda-1)x^{\lambda-2} + Bx^{r+1}\lambda x^{\lambda-1} + Cx^r\; x^\lambda = 0, \qquad (3)$$

or the same as

$$\mathrm{A}\,\lambda(\lambda-1) + B\,\lambda + C = 0. \qquad (4)$$

So let us note the characteristic equation of the equation (2) denoted by the formula:

$$\mathrm{P}(\lambda) = \mathrm{A}\,\lambda(\lambda-1) + B\,\lambda + C. \qquad (5)$$

Therefore for our investigation, we distinguish some cases.

For the beginning let us have:

1) λ_1, λ_2 −are simple different real roots of the equation (4) such that $\lambda_1 > -1, \lambda_2 > -1$. Then the solution of the equation (2) in the sense of *K'* is defined by the formula:

$$y(x) = c_1 x^{\lambda_1}\theta(x) + c_2|x|^{\lambda_1}\theta(-x) + c_3 x^{\lambda_2}\theta(x) + \ c_4|x|^{\lambda_2}\theta(-x) \qquad (6)$$

where $c_1, c_2, c_3, \mathrm{and} c_4$ are arbitrary constants and $\theta(x) -$ is the Heaviside test function.

Let us immediately verify that in the formula (6) each term is really the solution of the equation (2). Infact, when we substitute into the equation (2) the function $y(x) = u(x) = \ c_1 x^{\lambda_1}\theta(x)$, by virtue of the following property

$$\forall y \in K', \forall\varphi \in K, \left(y^{(k)}(x), \varphi(x)\right) = (-1)^k \left(y(x), \varphi^{(k)}(x)\right) \qquad (7),$$

and the notation

$$A\int_0^\infty x^{\lambda_1}\left(x^{r+2}\varphi(x)\right)''dx\ - B\int_0^\infty x^{\lambda_1}\left(x^{r+1}\varphi(x)\right)'dx + C\int_0^\infty x^{\lambda_1}\, x^r\ \varphi(x)dx = I, \qquad (8)$$

after integration by part and taking into account $\lambda_1 > -1$, we obtain : I=

$$\mathrm{A}x^{\lambda_1}\left(x^{r+2}\varphi(x)\right)'\Big|_0^\infty - \lambda_1 A\int_0^\infty x^{\lambda_1-1}\left(x^{r+2}\varphi(x)\right)'dx - Bx^{\lambda_1}\left(x^{r+1}\varphi(x)\right)\Big|_0^\infty + \lambda_1 B\int_0^\infty x^{\lambda_1-1}\, x^{r+1}\ \varphi(x)dx\ + C\int_0^\infty x^{\lambda_1+r}\ \varphi(x)dx. \qquad (9)$$

Once more integrating by parts we reach to the following result:

$$I = -\lambda_1 \mathrm{A}x^{\lambda_1-1}x^{r+2}\varphi(x)\Big|_0^\infty + \lambda_1(\lambda_1-1)\mathrm{A}\int_0^\infty x^{\lambda_1+r}\ \varphi(x)dx + B\lambda_1\int_0^\infty x^{\lambda_1+r}\ \ \varphi(x)dx\ + C\int_0^\infty x^{\lambda_1+r}\ \ \varphi(x)dx = \int_0^\infty P(\lambda_1)\, x^{\lambda_1+r}\ \varphi(x)dx = 0, \qquad (10)$$

that is what has been said.

Analogously in the same way, we proof that the function defined by the expression:

$y(x) = v(x) = \ c_2|x|^{\lambda_1}\theta(-x)$is solution of the equation (2) that is meaning, when substituting $v(x)$ in the formula (2) we arrive to:

$$\mathrm{A}\int_{-\infty}^{0}(-x)^{\lambda_1}(x^{\lambda_1+r}\ \varphi(x))''dx - B\int_{-\infty}^{0}(-x)^{\lambda_1}(x^{\lambda_1+r}\ \varphi(x))'dx + C\int_{-\infty}^{0}(-x)^{\lambda_1}x^r\ \varphi(x)dx = J. \quad (11)$$

It is not difficult to note that this case is similar to the previous so that it is clear $J = 0$. It is the same way, we can proof for the remaining functions in the formula (6).

2) Next, we consider the case when the roots of the characteristic equation (4)λ_1 and λ_2 are such that, $\lambda_1 \leq -1\ and \lambda_2 \leq -1$. In this case, for the homogeneous equation it may exist generalized solutions not zero-centered in the form of principal parts or their derivatives. Let consider, for example, the case when $\lambda_1,\lambda_1 \in \mathbb{Z}_-$. The solutions of the equation(2) are defined by the formula:

$$y\,(x) = \ c_1 p\left(\frac{1}{x^k}\right) + \ c_2 p\left(\frac{1}{x^l}\right), \quad (12)$$

where $k = -\lambda_1$; $l = -\lambda_2$ and $p\left(\frac{1}{x^k}\right) -$ the generalized function defined before in [20].

Let us verifiy that each term in the formula (12) is solution of the equation (2) i.e it is fulfilled the equality:

$$y\,(x) = Ax^{r+2}\left[p\left(\frac{1}{x^k}\right)\right]'' + Bx^{r+1}\left[p\left(\frac{1}{x^k}\right)\right]' + Cx^r p\left(\frac{1}{x^k}\right)0 \quad (13)$$

or the same as the following by the rule of differentiation:

$$Ax^{r+2}k(k+1)p\left(\frac{1}{x^{k+2}}\right) - Bx^{r+1}kp\left(\frac{1}{x^{k+1}}\right) + Cx^r p\left(\frac{1}{x^k}\right) = 0 \quad (14)$$

Next, we use the following proved properties in [20].

$$\forall\, m \in \mathbb{N}, \forall\, n \in \mathbb{N},\ x^m p\left(\frac{1}{x^n}\right) = \begin{cases} x^{m-n}, m \geq n \\ p\left(\frac{1}{x^{n-m}}\right), n > m. \end{cases} \quad (15)$$

Let in the beginning $r \geq k$. Then taking into account the property (15), we reach to the following result:

$$Ak(k+1)x^{r-k} - Bkx^{r-k} + Cx^{r-k} = x^{r-k}(Ak(k+1) - Bk + C).$$

If changing $k\ into\ -\lambda$ we will obtain:

$$x^{r+\lambda}(A\lambda(\lambda-1) + B\lambda + C) = x^{r+\lambda}\mathrm{P}(\lambda) = 0. \quad (16)$$

Now we consider the case, when $r < k$. Then taking into account the property (15) we have:

$$Ak(k+1)p\left(\frac{1}{x^{k-r}}\right) - Bkp\left(\frac{1}{x^{k-r}}\right) + Cp\left(\frac{1}{x^{k-r}}\right) = p\left(\frac{1}{x^{k-r}}\right)(A\lambda(\lambda-1) + B\lambda + C) = p\left(\frac{1}{x^{k-r}}\right)\mathrm{P}(\lambda) = 0. \quad (17)$$

So that we completely proved that it is taking place the property of the equality (13).

3) Now, let λ be a root of the characteristic polynomial of multiplicity 2 of $\mathrm{P}(\lambda)$ and $\lambda > -1$. Then the general solution of the equation (2) in the sense of *K'* is defined by the following formula:

$$y\,(x) = \ c_1 x^\lambda\theta(x) + \ c_2|x|^\lambda\theta(-x) + \ c_3 x^\lambda ln x\theta(x) + c_4\,|x|^\lambda\, ln|x|\theta(-x). \quad (18)$$

where $c_1, c_2, c_3,$ andc_4 are arbitrary constants and $\theta(x)$-the Heaviside Test function.

Let proof, that in this writing, each term separately is a solution of the equation (2). Note that for the two first terms, the proof have already been done so that it remains to proof that

$$y\,(x) = u(x) = \ c_3 x^\lambda ln x\theta(x)\ and \qquad y\,(x) = v(x) = \ c_4\,|x|^\lambda ln|x|\theta(-x) \quad (19).$$

separately are solutions of the equation (2).

We begin with the expression

$$y\,(x) = u(x) = \ c_3 x^\lambda ln x\theta(x) \quad (20).$$

Setting(20) into (2) gives us the following result for every $\varphi(x) \in K$.

$$\left(\mathrm{A}x^{r+2}\,y''(x) + Bx^{r+1}y'(x) + Cx^r\ y(x), \varphi(x)\right) = \mathrm{A}\int_0^{+\infty} x^\lambda\, ln x\left(x^{r+2}\varphi(x)\right)''dx - B\int_0^{+\infty} x^\lambda\, ln x\left(x^{r+1}\varphi(x)\right)'dx + C\int_0^{+\infty} x^r\, x^\lambda ln x\varphi(x)dx \quad (21).$$

Now integrating part by part twice in the first integral and once in the second term, we reach to the following:

$$\mathrm{I} = \mathrm{A}x^\lambda ln x\left(x^{r+2}\varphi(x)\right)'\Big|_0^{+\infty} - A\int_0^{+\infty} x^{\lambda-1}\,(\lambda ln x + 1)\left(x^{r+2}\varphi(x)\right)'dx - Bx^\lambda ln x\left(x^{r+1}\varphi(x)\right)\Big|_0^{+\infty} + B\int_0^{+\infty} x^{\lambda-1}\,(\lambda ln x + 1)x^{r+1}\varphi(x)dx + \ C\int_0^{\infty} x^{r+\lambda}\,\varphi(x)dx \quad (22).$$

Note that

$$x^\lambda ln x\left(x^{r+2}\varphi(x)\right)'\Big|_0^{+\infty} = 0$$

and also (23)

$$x^\lambda ln x\left(x^{r+1}\varphi(x)\right)\Big|_0^{+\infty} = 0.$$

Therefore, it is true that

$$I = -\mathrm{A}x^{\lambda-1}(\lambda ln x + 1)\left(x^{r+2}\varphi(x)\right)\Big|_0^{+\infty} + A\int_0^{+\infty} x^{\lambda-2}\,[\lambda + (\lambda-1)(\lambda ln x + 1)]x^{r+2}\varphi(x)dx + B\int_0^{+\infty} x^{\lambda+r}\,(\lambda ln x + 1)\varphi(x)dx + C\int_0^{+\infty} x^{r+\lambda}\,\varphi(x)ln x dx = \int_0^{+\infty} x^{\lambda+r}\left[A\left(\lambda + (\lambda-1)\right) + B\right]\varphi(x)dx + \int_0^{+\infty} x^{\lambda+r}\,[A\lambda(\lambda-1) + B\lambda + C]\varphi(x)ln x dx. \quad (24).$$

We recall that the second integral is equal to zero by virtue of $A\lambda(\lambda-1) + B\lambda + C = \mathrm{P}(\lambda) = 0$. It remains to show that

$$I = \int_0^{+\infty} x^{\lambda+r}\,[A(2\lambda-1) + B]\varphi(x)dx = 0 \quad (25)$$

Or writing in another way we obtain

$$I = \int_0^{+\infty} [A(2\lambda-1) + B](x^{\lambda+r}\varphi(x))\,dx = 0 \quad (26)$$

As λ is a multiplicity root of two of$\mathrm{P}(\lambda)$, so the discriminant of the characteristic equation

$$\mathrm{P}(\lambda) = A\lambda^2 + (B-A)\lambda + C = 0$$

has the following form.

$$\Delta\mathrm{P}(\lambda) = (B-A)^2 - 4\mathrm{AC} = 0, \quad (27)$$

from where $\lambda_1 = \lambda_2 = \lambda = \frac{A-B}{2A}$ (28)

Next, setting λ in the formula (26), we have

$$I = \int_0^{+\infty}\left[A\left(2\frac{A-B}{2A} - 1\right) + B\right](x^{\lambda+r}\varphi(x))\,dx \quad (29).$$

Simple calculations will achieves the proof, so that finally $I = 0$. Analogously it is proved that the function

$$y\ (x) = v(x) = \ c_4\,|x|^{\lambda}\,ln|x|\theta(-x) \qquad (30).$$

is the solution of equation (2), so it is true the following expression

$$J = A\int_{-\infty}^{0}(-x)^{\lambda}\,ln(-x)\big(x^{r+2}\varphi(x)\big)''dx\ - B\int_{-\infty}^{0}(-x)^{\lambda}\,ln(-x)\big(x^{r+1}\varphi(x)\big)'dx + C\int_{-\infty}^{0}(-x)^{\lambda}\,ln(-x)x^{r}\varphi(x)dx = 0 \qquad (31)$$

Now integrating twice part by part the first term and once the second term we obtain:

$$J = \mathrm{A}(-x)^{\lambda}ln(-x)\big(x^{r+2}\varphi(x)\big)'\Big|_{-\infty}^{0} - A\int_{-\infty}^{0}(-x)^{\lambda-1}\,(-\lambda ln(-x)+1)\big(x^{r+2}\varphi(x)\big)'dx - B(-x)^{\lambda}ln(-x)\big(x^{r+1}\varphi(x)\big)\Big|_{-\infty}^{0} + B\int_{-\infty}^{0}(-x)^{\lambda-1}\,(-\lambda ln(-x)+1)x^{r+1}\varphi(x)dx + C\int_{-\infty}^{0}(-x)^{\lambda}\,ln(-x)x^{r}\varphi(x)dx. \qquad (32)$$

We can also proof that it is true the following relationship:

$$(-x)^{\lambda}ln(-x)\big(x^{r+2}\varphi(x)\big)'\Big|_{-\infty}^{0} = 0$$

and also (33)

$$(-x)^{\lambda}ln(-x)\big(x^{r+1}\varphi(x)\big)\Big|_{-\infty}^{0} = 0.$$

Therefore by virtue of all what have been said and done we have:

$$J = -\mathrm{A}(-x)^{\lambda-1}(-\lambda ln(-x)+1)x^{r+2}\varphi(x)\Big|_{-\infty}^{0} + A\int_{-\infty}^{0}(-x)^{\lambda-2}\,[-(\lambda-1)(-\lambda ln(-x)+1)-\lambda]x^{r+2}\varphi(x)dx + B\int_{-\infty}^{0}(-x)^{\lambda-1}\,x^{r+1}(-\lambda ln(-x)+1)\varphi(x)dx + C\int_{-\infty}^{0}(-x)^{\lambda}\,x^{r}ln(-x)\varphi(x)dx. \qquad (34)$$

Now grouping the sub-integral term in the following way we obtain:

$$J = A\int_{-\infty}^{0}(-x)^{\lambda}\,x^{r}[\lambda(\lambda-1)A + B\lambda + C]ln(-x)\varphi(x)dx + \int_{-\infty}^{0}(-x)^{\lambda}\,x^{r}[A(1-2\lambda)-B]\varphi(x)dx = \int_{-\infty}^{0}(-x)^{\lambda}x^{r}P(\lambda)\,\varphi(x)ln(-x)dx + \int_{-\infty}^{0}(-x)^{\lambda}x^{r}\left[A\left(1-2\frac{A-B}{2A}\right)-B\right]\varphi(x)dx. \qquad (35).$$

Simple calculations will end the proof, so that finally we reach $J = 0$. Therefore, we have shown that each term in the formula (18) is a solution of the equation (2).

4. Now let the roots of the characteristic equation $\mathrm{P}(\lambda)$ are complex conjugate, i.e

$$\lambda_1 = \alpha + i\beta \quad and\ \ \lambda_2 = \alpha - i\beta, \qquad (36).$$

Then the solution of the equation (2) in the sense of *K'* has the following form:

$$y\ (x) = c_1x^{\alpha}\cos(\beta lnx)\,\theta(x) + c_2|x|^{\alpha}\cos(\beta ln|x|)\,\theta(-x) + c_3x^{\alpha}\sin(\beta lnx)\,\theta(x) + c_4|x|^{\alpha}\sin(\beta ln|x|)\,\theta(-x) \qquad (37)$$

Let proof that each term in this expression is a solution of the equation (2).

Let beginning with the function

$$y\ (x) = u(x) = c_1x^{\alpha}\cos(\beta lnx)\,\theta(x) \qquad (38)$$

The setting of the function $y\ (x)$ gives us

$$(\mathrm{A}x^{r+2}\,y''(x) + Bx^{r+1}y'(x) + Cx^{r}\ y(x), \varphi(x)) \ = A\int_{0}^{+\infty}x^{\alpha}\cos(\beta lnx)\,\big(x^{r+2}\varphi(x)\big)''dx - B\int_{0}^{+\infty}x^{\alpha}\cos(\beta lnx)\,\big(x^{r+1}\varphi(x)\big)'dx + C\int_{0}^{+\infty}x^{\alpha}\cos(\beta lnx)x^{r}\varphi(x)dx\ . \qquad (39)$$

Next, integrating twice in the first term and once in the second term, we have:

$$I = \mathrm{A}x^{\alpha}\cos(\beta lnx)\,\big(x^{r+2}\varphi(x)\big)'\Big|_{0}^{+\infty} - A\int_{0}^{+\infty}x^{\alpha-1}(\alpha\cos(\beta lnx) - \beta sin(\beta lnx)\big(x^{r+2}\varphi(x)\big)'dx - Bx^{\alpha}\cos(\beta lnx)\,\big(x^{r+1}\varphi(x)\big)\Big|_{0}^{+\infty} + B\int_{0}^{+\infty}x^{r+1}\varphi(x)[x^{\alpha-1}(\alpha\cos(\beta lnx) - \beta sin\beta lnx)]\,dx + C\int_{0}^{+\infty}x^{\alpha+r}\cos(\beta lnx)\varphi(x)dx. \qquad (40)$$

It is easy to see that it is taking place the following relations:

$$\mathrm{A}x^{\alpha}\cos(\beta lnx)\,\big(x^{r+2}\varphi(x)\big)'\Big|_{0}^{+\infty} = 0$$

and also (41)

$Bx^{\alpha}\cos(\beta lnx)\,\big(x^{r+2}\varphi(x)\big)'\Big|_{0}^{+\infty}$=0

Therefore integrating once more the first term, we reach to the following result.

$$I = -\mathrm{A}x^{\alpha-1}\,(\alpha\cos(\beta lnx) - \beta sin(\beta lnx)\,\big(x^{r+2}\varphi(x)\big)\Big|_{0}^{+\infty} + A\int_{0}^{+\infty}x^{r+2}\varphi(x)\left[x^{\alpha-2}(\alpha-1)(\alpha\cos(\beta lnx) - \beta sin(\beta lnx)) + x^{\alpha-1}(-\tfrac{1}{x}\alpha\beta sin\beta lnx - \tfrac{1}{x}\beta^2\cos(\beta lnx))\right]dx + B\int_{0}^{+\infty}x^{r+1}\varphi(x)[\alpha x^{\alpha-1}(\cos(\beta lnx) - x^{\alpha-1}\beta sin\beta lnx)]\,dx + C\int_{0}^{+\infty}x^{\alpha}\cos(\beta lnx)x^{r}\varphi(x)dx \qquad (42)$$

After opening the parentheses and grouping together we obtain:

$$\begin{aligned} I = A\int_{0}^{+\infty} x^{r+2}\varphi(x)x^{\alpha-2}[\alpha(\alpha &- 1)\cos(\beta lnx) - \alpha\beta sin(\beta lnx) \\ &+ \beta sin(\beta lnx) - \alpha\beta sin(\beta lnx \\ &- \beta^2\cos(\beta lnx)]dx \\ &+ B\int_{0}^{+\infty}(x^{\alpha+r}\alpha\cos(\beta lnx))\,\varphi(x) \\ &- x^{\alpha+r}\varphi(x)\beta sin(\beta lnx))\,dx \\ &+ C\int_{0}^{+\infty}x^{\alpha+r}\,\varphi(x)\cos(\beta lnx)dx \\ &= \int_{0}^{+\infty}x^{r+\alpha}\varphi(x)\cos(\beta lnx)\,[A\alpha(\alpha-1) \\ &+ B\alpha + C - A\beta^2]dx \\ &+ \int_{0}^{+\infty}x^{r+\alpha}\varphi(x)\,sin(\beta lnx)\,[-2\alpha\beta A + A\beta \\ &- B\beta]dx \end{aligned} \qquad (43)$$

We recall that λ is a root of the characteristic equation $\mathrm{P}(\lambda)$, and more λ is a complex conjugate one. Therefore, from the fact that

$\mathrm{P}(\lambda) = \mathrm{P}(\alpha + i\beta) = 0$, it follows

$$A(\alpha+i\beta)(\alpha-1+i\beta) + \mathrm{B}(\alpha+i\beta) + \mathrm{C} = 0, \qquad (44)$$

from where

$A[\alpha(\alpha-1)+i\alpha\beta+i\alpha\beta-i\beta-\beta^2]+B\alpha+iB\beta+C=0$ (45),

Or that is the same as:

$A\alpha(\alpha-1)+B\alpha+C-\beta^2A+i(2A\alpha\beta-A\beta+B\beta)=0.$ (46)

Next, we have at the same time

$$\begin{cases} A\alpha(\alpha-1)+B\alpha+C-\beta^2A=0 \\ 2A\alpha\beta-A\beta+B\beta=0 \end{cases} \quad (47)$$

With consideration of these two conditions we obtain the needed, so that it is proved $I=0$.

Now let consider the second case when the root of the characteristic equation, $\lambda=\alpha-i\beta$. In this case $\mathrm{P}(\lambda)=\mathrm{P}(\alpha-i\beta)=0$ is equivalent to the following

$A(\alpha-i\beta)(\alpha-1-i\beta)+B(\alpha-i\beta)+C=0,$

from where it follows once more the condition (47). That is given the right to us writing in this case $I=0$. Analogously it can be proved that the term

$$y(x)=v(x)=\ c_3x^\alpha sin(\beta lnx)\theta(x) \quad (48)$$

is also the solution of the equation (2), that is meaning $\forall\varphi(x)\in\ K$ it is true:

$$J = A\int_0^{+\infty}x^\alpha\, sin(\beta lnx)\left(x^{r+2}\varphi(x)\right)''dx - B\int_0^{+\infty}x^\alpha\, sin(\beta lnx)\left(x^{r+1}\varphi(x)\right)'dx + C\int_0^{+\infty}x^\alpha\, sin(\beta lnx)\left(x^r\varphi(x)\right)dx=0. \quad (49)$$

Integrating by parts twice the first term and once the second term, we have:

$$J = \mathrm{A}x^\alpha \sin(\beta lnx)\left(x^{r+2}\varphi(x)\right)'\Big|_0^{+\infty} - A\int_0^{+\infty}\left(x^{r+2}\varphi(x)\right)'[\alpha x^{\alpha-1}\,sin(\beta lnx)+\beta x^{\alpha-1}\cos(\beta lnx)]dx\ - Bx^\alpha\sin(\beta lnx)\left(x^{r+1}\varphi(x)\right)\Big|_0^{+\infty} + B\int_0^{+\infty}x^{r+1}\varphi(x)[\alpha x^{\alpha-1}\,sin(\beta lnx)+\beta x^{\alpha-1}cos(\beta lnx)]\,dx + C\int_0^{+\infty}x^{\alpha+r}\varphi(x)\sin(\beta lnx)dx. \quad (50)$$

As $\forall\varphi(x)\in\ K$, then it is accomplished the following relations:

$$x^\alpha\sin(\beta lnx)\left(x^{r+2}\varphi(x)\right)'\Big|_0^{+\infty}=0,$$

and also (51)

$$x^\alpha\sin(\beta lnx)\left(x^{r+1}\varphi(x)\right)\Big|_0^{+\infty}=0.$$

Therefore,

$$J=-Ax^{r+2}\varphi(x)[\alpha x^{\alpha-1}(\alpha\, sin(\beta lnx)+\beta cos(\beta lnx))]\Big|_0^{+\infty} + A\int_0^{+\infty}x^{r+2}\varphi(x)[(\alpha-1)x^{\alpha-2}(\alpha sin(\beta lnx)+\beta cos(\beta lnx)+ x^{\alpha-2}(\alpha\beta cos(\beta lnx)-\beta^2 sin(\beta lnx)]\,dx + B\int_0^{+\infty}\alpha x^{r+\alpha}\varphi(x)\left(sin(\beta lnx)+\beta x^{r+\alpha}cos(\beta lnx)\right)dx + C\int_0^{+\infty}x^{\alpha+r}\varphi(x)\sin(\beta lnx)dx \quad (52)$$

Now, let opening brackets and grouping, we have:

$$J=\int_0^{+\infty}x^{r+\alpha}\varphi(x)\,sin(\beta lnx)\,[A\alpha(\alpha-1)-A\beta^2+B\alpha+\mathrm{C}]dx+\int_0^{+\infty}x^{r+\alpha}\varphi(x)\,cos(\beta lnx)\,[(\alpha-1)\beta A+A\alpha\beta+\mathrm{B}\beta]dx. \quad (53)$$

From there and from the previous considerations, we deduce the needed, so that, it is proved $J=0$.

Similarly, it can be proved that the remaining terms in the formula (37) are, separately, each one, solutions of the equation (2).

Now, on the basis of all what have been done and considering our previous results obtained and related to the non homogeneous equation of the same type with Dirac delta function in the right hand side, we can definitively write the general solution of the equation:

$$\mathrm{A}x^m\, y''(x)+Bx^ny'(x)+Cx^r\, y(x)=\delta^{(s)}(x) \quad (1').$$

Theorem 3.1let $\mathrm{A}\cdot\mathrm{B}\cdot\mathrm{C}\neq 0, \mathrm{m},\mathrm{n}\in\mathbb{N},\ \mathrm{r},\mathrm{s}\in\mathbb{N}\cup\{0\}$ and fulfilled the conditionm $=\mathrm{r}+2, \mathrm{n}=\mathrm{r}+1, \mathrm{C}-\mathrm{B}(\mathrm{s}+\mathrm{r}+1)+\mathrm{A}(\mathrm{s}+\mathrm{r}+2)(\mathrm{s}+\mathrm{r}+1)\neq 0.$

Then, the general solution of the equation (1') in the space K' has the following form:

1) Let $\forall\, j\in\mathbb{Z}_+, C-B(j+r+1)+A(j+r+2)(j+r+1)\neq 0.$

and the roots of the characteristic equation $\mathrm{P}(\lambda)\lambda_1,\lambda_2$ simples and different, such that $\lambda_1>-1\ \ and\lambda_2>-1$

$$y(x) = \frac{(-1)^r s!}{(s+r)!\,[C-B(s+r+1)+A(s+r+2)(s+r+1)]}\delta(x)^{(s+r)} + \sum_{j=0}^{r-1}C_j\ \delta(x)^{(j)}+C'\ x^{\lambda_1}\theta(x)+C'_2|x|^{\lambda_1}\theta(-x)+C'_3x^{\lambda_2}\theta(x) + C'_4|x|^{\lambda_2}\theta(-x), \quad (54)$$

where $C_0,\ldots\ldots.C_{r-1}, C'_1, C'_2, C'_3\ and\ C'_4$ are arbitrary constants.

2) Let existing at least one $j_*^1\in\mathbb{Z}_+/\{s\}$ or $j_*^2\in\mathbb{Z}_+/\{s\}$ such that, $\mathrm{P}_2(j_*^i)=0\,, i=1{,}2$ and the roots $\lambda_1,\lambda_2\in\mathrm{P}(\lambda)\ such\ that\ \lambda_1>-1,\lambda_2>-1$, then the solution is defined by the following formula:

$$y(x)=\frac{(-1)^r s!}{(s+r)![C-B(s+r+1)+A(s+r+2)(s+r+1)]}\delta(x)^{(s+r)}+\sum_{j=0}^{r-1}C_j\ \delta(x)^{(j)}\ +\ \sum_{j_*^i\in\, Nul\, P_2(j)}\ C_{j_*^i+r}\,\delta(x)^{(j_*^m+r)}+C'_1x^{\lambda_1}\theta(x)+C'_2|x|^{\lambda_1}\theta(-x)+C'_3x^{\lambda_2}\theta(x)+C'_4|x|^{\lambda_2}\theta(-x), \quad (55)$$

where $C_0,\ldots\ldots.C_{r-1}, C'_1, C'_2, C'_3, C'_4\ and C_{j_*^i+r}(i=1{,}2)$ are arbitrary constants.

3) Let $\forall\, j\in\mathbb{Z}_+, C-B(j+r+1)+A(j+r+2)(j+r+1)\neq 0\ and\ \mathrm{P}(\lambda)$ having roots of multiplicity two: then the solution is defined by the following formula:

$$y(x) = \frac{(-1)^r s!}{(s+r)!\,[C-B(s+r+1)+A(s+r+2)(s+r+1)]}\delta(x)^{(s+r)} + \sum_{j=0}^{r-1}C_j\ \delta(x)^{(j)}+C'\ x^\lambda\theta(x)+C'_2|x|^\lambda\theta(-x) + C'_3x^\lambda lnx\theta(x)+C'_4|x|^\lambda ln|x|\theta(-x), \quad (56)$$

where $C_0,\ldots\ldots.C_{r-1}, C'_1, C'_2, C'_3\ and\ C'_4$ are arbitrary constants.

4) Let existing at least one $j_*^1\in\mathbb{Z}_+/\{s\}$ or $j_*^2\in\mathbb{Z}_+/\{s\}$ such that , $\mathrm{P}_2(j_*^i)=0\,, i=1{,}2$and more λ is a root of multiplicity two of the characteristic polynom $\mathrm{P}(\lambda)$. Then

$y(x) = \frac{(-1)^r s!}{(s+r)![C-B(s+r+1)+A(s+r+2)(s+r+1)]}\delta(x)^{(s+r)} + \sum_{j=0}^{r-1} C_j\ \delta(x)^{(j)} + \sum_{j_*^i \in Nul\, P_2(j)} C_{j_*^i+r}\,\delta(x)^{(j_*^i+r)} + C'_1 x^\lambda \theta(x) + C'_2|x|^\lambda \theta(-x) + C'_3 x^\lambda lnx\theta(x) + C'_4|x|^\lambda ln|x|\theta(-x),$ (57)

where $C_0, \ldots\ldots. C_{r-1}, C'_1, C'_2, C'_3, C'_4\ and C_{j_*^i+r} (i = 1,2)$ are arbitrary constants.

5) Let $\forall j \in \mathbb{Z}_+, C - B(j + r + 1) + A(j + r + 2)(j + r + 1) \neq 0$ and $\lambda_1, \lambda_2 \in \mathrm{P}(\lambda)$ such that $\lambda_1, \lambda_2 \in \mathbb{Z}_-$; $\lambda_1 \neq \lambda_2$. Then $y(x) = \frac{(-1)^r s!}{(s+r)![C-B(s+r+1)+A(s+r+2)(s+r+1)]}\delta(x)^{(s+r)} + \sum_{j=0}^{r-1} C_j\ \delta(x)^{(j)} + C'_1\, p\left(\frac{1}{x^{-\lambda_1}}\right) + C'_2\, p\left(\frac{1}{x^{-\lambda_2}}\right),$ (58)

where $C_0, \ldots\ldots. C_{r-1}, C'_1, and\ C'_2$are arbitrary constants.

6) Let existing at least one $j_*^1 \in \mathbb{Z}_+ /\{s\}$ or $j_*^2 \in \mathbb{Z}_+ /\{s\}$ such that , $\mathrm{P}_2(j_*^i) = 0, i = 1,2$ and more$\lambda_1, \lambda_2 \in \mathrm{P}(\lambda)$such that $\lambda_1, \lambda_2 \in \mathbb{Z}_-$;$\lambda_1 \neq \lambda_2$. Then $y(x) = \frac{(-1)^r s!}{(s+r)![C-B(s+r+1)+A(s+r+2)(s+r+1)]}\delta(x)^{(s+r)} + \sum_{j=0}^{r-1} C_j\ \delta(x)^{(j)} + \sum_{j_*^i \in Nul\, P_2(j)} C_{j_*^i+r}\,\delta(x)^{(j_*^i+r)} + C'_1\, p\left(\frac{1}{x^{-\lambda_1}}\right) + C'_2\, p\left(\frac{1}{x^{-\lambda_2}}\right),$ (59)

where $C_0, \ldots\ldots. C_{r-1}, C'_1, C'_2, and C_{j_*^i+r}\ (i = 1,2)$ are arbitrary constants.

7) Let $\forall j \in \mathbb{Z}_+, C - B(j + r + 1) + A(j + r + 2)(j + r + 1) \neq 0$ and λ_1, λ_2 complex conjugate roots of $\mathrm{P}(\lambda)$, where $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$. Then the general solution is defined by the formula

$y(x) = \frac{(-1)^r s!}{(s+r)![C-B(s+r+1)+A(s+r+2)(s+r+1)]}\delta(x)^{(s+r)} + \sum_{j=0}^{r-1} C_j\ \delta(x)^{(j)} + C'_1\, x^\alpha cos(\beta lnx)\theta(x) + C'_2|x|^\alpha cos(\beta ln|x|)\theta(-x)) + C'_3 x^\alpha sin(\beta lnx)\theta(x) + C'_4\, |x|^\alpha sin(\beta ln|x|)\theta(-x)),$ (60)

where $C_0, \ldots\ldots. C_{r-1}, C'_1, C'_2, C'_3, and\ C'_4$ are arbitrary constants.

8) Let existing at least one $j_*^1 \in \mathbb{Z}_+ /\{s\}$ or $j_*^2 \in \mathbb{Z}_+ /\{s\}$ such that,$\mathrm{P}(j_*^i)_2 = 0, i = 1,2$ and moreλ_1, λ_2 complex conjugate roots of $\mathrm{P}(\lambda)$. Then the general solution of the equation (1') is defined by the formula:

$y(x) = \frac{(-1)^r s!}{(s+r)![C-B(s+r+1)+A(s+r+2)(s+r+1)]}\delta(x)^{(s+r)} + \sum_{j=0}^{r-1} C_j\ \delta(x)^{(j)} + \sum_{j_*^i \in Nul\, P_2(j)} C_{j_*^i+r}\,\delta(x)^{(j_*^m+r)} + C'_1\, x^\alpha cos(\beta lnx)\theta(x) + C'_2|x|^\alpha cos(\beta ln|x|\theta(-x)) + C'_3 x^\alpha sin(\beta lnx)\theta(x) + C'_4\, |x|^\alpha sin(\beta ln|x|\theta(-x)),$ (61)

where $C_0, \ldots\ldots. C_{r-1}, C_{j_*^i+r}(i = 1,2), C'_1, C'_2, C'_3, and\ C'_4$ are arbitrary constants.

Remark 1. In this section we have just write only the general solution of the more simple equation (1') in the situation called *Euler case*. Therefore, from the same ideas we can also be able to find the general solution of a more general equation in such situation of the form $\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = \delta(x)^{(s)}$(homogeneous and non homogeneous equations). Before concluding, let say that the function $y(x) \in K'$ is solution of the differential equations (1)-(2) in the sense of distributionsif and only if :

$$\forall\varphi(x) \in K$$
$$(\mathrm{A}x^{r+2}\, y''(x) + Bx^{r+1}y'(x) + Cx^r y(x), \varphi(x)) = (0, \varphi(x)) = 0.$$

Now let us state a last following remark.

Remark 2. Therefore, we understand and imagine already the huge calculations to be made when substituting the formula of the general solutions obtained upstair, into the initial equation to verify the needed equality.

CONCLUSION

In this paper we have completely investigated the existence and the analysis of classical solutions of a second-*order* linear singular homogeneous differential equation in the *Euler case situation* in the space of generalized functions K'. We have look for the solution by replacing the form of the particular solution in the form of $y(x) = x^\lambda$ (where λ- unknown value) into the considered equation. The obtained characteristic polynomial $P(\lambda)$ guided us, depending of the nature of the roots to write the wanted general solution of the equation (1). All the results obtained within the investigation are formulated in the global theorem 3.1, which describe , case by case, the general solution of the non homogeneous equation$\mathrm{A}x^m\, y''(x) + Bx^n y'(x) + Cx^r\, y(x) = \delta^{(s)}(x)$ having the same homogeneous equation of the type (1).

RECOMMENDATIONS

This achieved work will help us to undertake in a brief future, the investigation of classical solutions of a more general standard situation of *l-order* linear singular homogeneous differential equation in the space of generalized function K' of the following type$\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = 0$, when the parameters $k_i$satisfy the conditions$k_i = k_0 + i;\ i = 0,1\ldots\ldots, l$ and, for which we have already defined all the generalized - function solutions in the space K'. See our recent publication [16]. Therefore and consequently, we could also completely may be able to describe both all the generalized-functions and classical regular solutions of the equation evoked.

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