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ON A FIRST ORDER LINEAR SINGULAR DIFFERENTIAL EQUATION IN THE SPACE OF GENERALIZED FUNCTIONS $(S_0^\beta)'$.

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ABSTRACT

We investigate the question of the appearance of other kind of solutions of an homogeneous singular linear first -order differential equation not found in the space K' but existing in the space of rapidly decreasing generalized functions $(S_0^\beta)'$. Previously in our work, we have already obtained all distributional solutions of the considered equation in the space K'. We want to show in the present investigation the possibility of appearance of other kind of solutions in the space $(S_0^\beta)'$. For this goal, we use the well known theorems and lemmas from the general theory of generalized functions and theory of linear differential equations. We look for the particular solution in a special chosen way and we formulate step by step the corresponding theorems related to each case for the description of the not zero-centered general solution.

Keywords: Rapidly decreasing function, tempered distribution, not zero-centered solution, singular linear differential equation.

2020 Mathematics Subject Classification: 34A05, 39A05, 42B10, 44A05.

INTRODUCTION

We recall that from the general theory of differential equations, the expression of the following form $\sum_{i=0}^{n} c_i b_i(t) y^{(i)}(t) = f(t)$, where the variables coefficients $c_i b_i(t), i = 0,1,\dots n, f(t)$ are continuous real functions with $b_n(t) \neq 0, c_i -$ arbitrary constants is called an $n-order$ linear differential equation. When investigating a simple case of $nth-order$ linear differential equation, some elements of the construction of the solutions (classical regular and distributional) can be found in our recent works in the case of the space K'. See [23,24,25,27]. Many authors devoted various scientific works to some specific classes of differential equations investigated for the existence of solutions in some special functional spaces. It is well known that various methods were developped in finding distributional solutions and, in the case of a simple $4\text{th}-\text{order}$ Euler differential equation of the form $t^4 y^{(4)}(t) + t^3 y'''(t) + t^2 y''(t) + t y'(t) + m y(t) = 0$ where m is an integer and $t \in \mathbb{R}$ studied by the authors Krall et al, Little John and Kanwall, they obtained results which were confirmed in the researches conducted by Amphon Liangprom, Kamsing Nonlaopon. For more details on the application of Laplace technique used by them see [10, 26, 28]. All what has been said lead us to state that, not depending of methods developped to solve a differential equation in some specific functional space of solutions, the results are always the same and when changing the space in which the investigations for solvability are conducted, we can reach to other forms of the solutions. On the basis of what has been said upstair, we undertake in this work research of other kind of solutions in the space $(S_0^\beta)'$ of a singular first order linear differential equation already investigated in the space *K'*. Namely, in this work we investigate the question of the appearence of solutions of the other nature in the space of generalized functions $(S_0^\beta)'$ for the homogeneous linear singular differential equation of the first order defined by the following formula

$$Ax^p y'(x) + Bx^q y(x) = 0 \qquad (1),$$

Where A, B are real numbers and $p \in \mathbb{N}, q \in \mathbb{N}\{0\}, y(x) \in (S_0^\beta)'$. We recall that such equation has been investigated in a more small space of generalized functions *K'. See* [23,24, 25, 27]. We undertake investigations in solving this equation in a more bigger space of generalized functions$(S_0^\beta)'$ in which we arrive to show the existence of other solutions than those found in *K'.* This paper is structured as follows: in section 2, we recall some fundamental well known concepts of test functions and distributions (Rapidly decreasing function, tempered distribution,generalized functions from the spaceS'$(\mathbb{R}^n)$). Section 3 properly is presenting the investigation of the solvability (existence of not zero-centered solutions of the considered equation in all the situations connected with the relationship between the parameters A, B, p and q. We summarize and conclude our paper insection 4 followed by some recommendations for the follow-up or future scientific works to undertake, stated in section 5.

PRELIMINARIES

Before proceeding to the statement of our main results, the following definitions and concepts from the theory of distributions are required. For more details we can also refeer to [1,7,15,16, 17, 18, 19, 20, 21] from where we take these notions. **SpaceS**$(\mathbb{R}^n)$.

Definition 2.1

We say that a function $C^\infty, f: \mathbb{R}^n \to \mathbb{C}$ is rapidly decreasing if $\sup|x^\alpha D^\beta f(x)| = 0 \;\; \forall \alpha, \beta \in \mathbb{N}^n$ or$\lim_{|x|\to\infty} |x^\alpha D^\beta f(x)| =$

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$0, \forall \alpha, \beta \in \mathbb{N}^n$ or $\sup_{|\beta|<m} \sup_{x\in\mathbb{R}^n} ((1+|x|^2)^m)\left|D^\beta f(x)\right| < +\infty, \forall \beta \in \mathbb{N}^n$.

Definition 2.2

We denote through $S(\mathbb{R}^m)$ the space of rapidly decreasing C^∞ functions. The topology of $S(\mathbb{R}^m)$ is defined by the family of semi-norms:

$$p_{k,m}(f) = \sup_{x\in\mathbb{R}^n, |\alpha|\le m} \left((1+|x|^2)^k\right)|D^\alpha f(x)| < +\infty.$$

Proposal 2.1

$D(\mathbb{R}^n) \subset S(\mathbb{R}^m) \subset \varepsilon(\mathbb{R}^n)$. The inclusions being continuous and with imagedense i.e $D \hookrightarrow S \hookrightarrow \varepsilon$.

A) Space S'($\mathbb{R}^n$).

It has been shown that $D \hookrightarrow S$, dense image continuous inclusion. Hence any distribution continuous for the topology induced by S on D can be extended into a formcontinuous linear on S.

Definition 2.3

We denote by $S'(\mathbb{R}^n)$ the space of distributions S on D.
$S'(\mathbb{R}^n)$ is called space of tempered distributions. We identify $S'(\mathbb{R}^n)$to the space of continuous linear forms on $S(\mathbb{R}^n)$.

Property 2.1

1. The derivative of a tempered distribution is tempered.
2. The product of a tempered distribution by a polynomial is a linear map contineous from S into S, hence from D intoD for the topology of S.
3. In general $S'(\mathbb{N})$ is a module over the ring of polynomials.

For our convenience we will use the following definition for the space of test functions $S_\alpha^\beta (\alpha \ge 0, \beta \ge 0)$.

Definition 2.4

Let $S_\alpha^\beta (\alpha \ge 0, \beta \ge 0)$ be the space consisting of infinitively differentiable functions $\varphi(x)(-\infty < x < \infty)$ satisfying the inequality

$$\left|x^\alpha \varphi^{(k)}(x)\right| \le CA^k B^q k^{k\alpha} q^{q\beta},$$

where constant A, B, C depending of the function φ .
In particular when $\alpha = 0$, function $\varphi(x)$ from the space S_0^β verify the inequalities

$$\left|x^\alpha \varphi^{(k)}(x)\right| \le CA^k B^q q^{q\beta}.$$

The space of generalized functions over the space S_0^β is denoted $(S_0^\beta)'$.

Now let us move to our main results of the investigation undertook in the following section.

MAIN RESULTS

The analysis of the proofs of theorems 3.3 and 3.4 from our previous work where $q \neq p-1$, conduct us to note that here, there are possibilities of the existence of other solutions in the space of generalized functions $(S_0^\beta)'$. See [25].

From that, we can state for example the following theorems related to the two situations when $q \neq p-1 (q > p-1 \text{ or } q < p-1)$.

Theorem 3.1. Let $A.B \neq 0, p \in \mathbb{N}, q \in \mathbb{Z}_+$ and accomplished the condition $q > p-1$. The homogeneous equation (1) admit not centered at zero solution in the space of generalized functions $(S_0^\beta)'$ defined by the formula:

$$y(x) = \sum_{m=0}^{q-p} C_{m+p-1}\, \delta^{(m+p-1)}(x) + \sum_{m=0}^{q-p} C_{m+p-1} \sum_{m=0}^{q-p} \left(\frac{A}{B}(-1)^{p+1-q}\right)^l \cdot \prod_{s=0}^{l-1} \frac{[m+p+s(q-p+1)]!}{[m+q+s(q-p+1)]!} \delta^{(m+l(q-p+1)+p-1)}(x),$$

where $C_{m+p-1}, m = 0,1,2,\ldots\ldots, q-p$ are arbitrary constants.

Proof.

Not centered at. zero solution of the equation (1) we are looking in the following type:

$$y(x) = \sum_{m=0}^{\infty} C_j\, \delta^{(j)}(x), \quad (2)$$

where C_j are unknown coefficients. Putting (2) into (1) we obtain the following:

$$Ax^p \sum_{j=0}^{\infty} c_j \delta^{(j+1)}(x) + Bx^q \sum_{j=0}^{\infty} c_j \delta^{(j)}(x) = 0 \quad (3)$$

After computations and changing the sum indices we have:

$$A \sum_{k=0}^{\infty} C_{k+p-1} \frac{(-1)^p (k+p)!}{k!} \delta^{(k)}(x) + B \sum_{j=q}^{\infty} C_{k+q} \frac{(-1)^q (k+q)!}{k!} \delta^{(k)}(x) = 0 \quad (4)$$

The previous conduct us to an infinity linear homogeneous algebraic system for the unknown coefficients C_j to be found of the following type:

$$C_{k+p-1} A \frac{(-1)^p (k+p)!}{k!} + C_{k+q} B \frac{(-1)^q (k+q)!}{k!} = 0, \quad k = 0,1,2,\ldots.. \quad (5)$$

As we are investigating the case when $q > p-1$, then for all $k = m + l(q-p+1)$ we can find C_{k+q} towards C_{k+p-1}, where $l = 0,1,2,\ldots\ldots;\ m = 0,1,2,\ldots., q-p$. From (5) we have the following relationship when $k = m + l(q-p+1)$.

$$C_{m+(l+1)(q-p+1)+p-1} = (-1)^{p+1-q} \frac{A}{B} \frac{[m+p+s(q-p+1)]!}{[m+q+s(q-p+1)]!} C_{m+p-1+l(q-p+1)}, \quad (6)$$

Further reccurently we can easily obtain the final result to express the coefficient $C_{m+(l+1)(q-p+1)+p-1}$ towards C_{m+p-1} by the following formula :

$$C_{m+(l+1)(q-p+1)+p-1} = \left(\frac{A}{B}(-1)^{p+1-q}\right)^l \prod_{s=0}^{l-1} \frac{[m+p+s(q-p+1)]!}{[m+q+s(q-p+1)]!} C_{m+p-1}, \quad (7)$$

for all $l = 0,1,2,\ldots..,$ and any fixed $m = 0,1,2,\ldots.,q-p$. Therefore,

$$y(x) = \sum_{m=0}^{q-p} C_{m+p-1}\left\{\delta^{(m+p-1)}(x) + \sum_{l=1}^{\infty}\left(\frac{A}{B}(-1)^{p+1-q}\right)^{l}\delta^{(m+l(q-p+1)+p-1)}(x)\prod_{s=0}^{l-1}\frac{[m+p+s(q-p+1)]!}{[m+q+s(q-p+1)]!}\right\} \quad (8)$$

where C_{m+p-1}, $m = 0,1,2,\ldots.,q-p$ are arbitrary constants. The theorem is proved.

Analogously we can prove also the theorem in the case when it is realized the condition $q < p-1$, and namely it take place.

Theorem 3.2. Let $A.B \neq 0, p \in \mathbb{N}, q \in \mathbb{Z}_+$ and accomplished the condition $q < p-1$. Then the homogeneous equation (1) admit not centered at zero solution in the space of generalized functions $(S_0^{\beta})'$ defined by the formula:

$$y(x) = \sum_{m=0}^{q-p-2} C_{m+q}\,\delta^{(m+q)}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\sum_{l=1}^{\infty}\left(\frac{B}{A}(-1)^{p-q-1}\right)^{l} \cdot \prod_{s=0}^{l-1}\frac{[m+q+s(p-1-q)]!}{[m+p+s(p-1-q)]!}\delta^{(m+q+l(p-1-q))}(x),$$

where C_{m+q}, $m = 0,1,2,\ldots..,p-q-2$ are arbitrary constants.

Proof.

From (5) we have for all $k = m + l(p-1-q)$ where $l = 0,1,2,\ldots..,$ and for any fixed $m = 0,1,2,\ldots.,p-q-2$ the realization of the following relationship.

$$C_{m+l(q-p+1)+p-1} = (-1)^{q+1-q}\frac{B}{A}\frac{[m+l(p-1-q)+q]!}{[m+l(p-1-q)+p]!}C_{m+l(p-1-q)+q}. \quad (9)$$

Analougously reccurently it is easy to obtain the final result to express the relationship between the coefficient $C_{m+l(q-p+1)+p-1}$ and $C_{m+l(p-1-q)+q}$ defined by the following rule :

$$C_{m+l(q-p+1)+p-1} = \left((-1)^{q+1-q}\frac{B}{A}\right)^{l}\prod_{s=0}^{l-1}\frac{[m+q+s(p-1-q)]!}{[m+p+s(p-1-q)]!}C_{m+q} \quad (10)$$

where $l = 0,1,2,\ldots..,$ for every fixed $m = 0,1,2,\ldots.,p-q-2$. Therefore we obtain:

$$y(x) = \sum_{m=0}^{q-p-2} C_{m+q}\,\delta^{(m+q)}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\sum_{l=1}^{\infty}\left(\frac{B}{A}(-1)^{p-q-1}\right)^{l} \cdot \prod_{s=0}^{l-1}\frac{[m+q+s(p-1-q)]!}{[m+p+s(p-1-q)]!}\delta^{(m+q+l(p-1-q))}(x), \quad (11)$$

where $C_{m+q}, m = 0,1,2,\ldots..,p-q-2.$ are arbitrary coefficients. The theorem is proved.

Note that we can trust the absolute convergence of the obtained series. Finally combining al those theorems upstair, we can construct the general solutions of the quation (1) using also our previous results from [25].

Theorem 3.3. Let $A.B \neq 0, p \in \mathbb{N}, q \in \mathbb{Z}_+$ and accomplished the condition $q = p-1, B \neq A(q+s+1)$, then the general solution of the equation (1) in the space of generalized functions $(S_0^{\beta})'$ is defined by the following formula:

1) If $B - A(j+q+1) \neq 0,\ j \in \mathbb{Z}_+$ and $\frac{B}{A} < 1$

$$y(x) = \frac{(-1)^q s!}{(s+q)![B-A(s+q+1)]}\delta^{(q+s)}(x) + \sum_{j=0}^{q-1} c_j\delta^{(j)}(x) + k_1 x^{-\frac{B}{A}}\theta(x) + k_2|x|^{-\frac{B}{A}}\theta(-x), \quad (12)$$

where $c_0,\ldots..,c_{q-1},k_1,k_2$ are arbitrary constants, $\theta(x) -$ Heaviside step function.

2) If $\exists j_* \in \mathbb{Z}_+\setminus\{s\}$ such that, $B - A(j_*+q+1) = 0$ and $-\frac{B}{A} \leq -1$,

$$y(x) = \frac{(-1)^q s!}{(s+q)![B-A(s+q+1)]}\delta^{(q+s)}(x) + c_{j_*+q}\delta^{(j_*+q)}(x) + \sum_{j=0}^{q-1} c_j\,\delta^{(j)}(x) \quad (13)$$

where $A, B, c_0,\ldots..,c_{q-1},c_{j_*+q},k_1,k_2$ are arbitrary constants.

3) If $B - A(j+q+1) \neq 0,\ j \in \mathbb{Z}_+$ and $\frac{B}{A} \geq 1$

$$y(x) = \frac{(-1)^q s!}{(s+q)![B-A(s+q+1)]}\delta^{(q+s)}(x) + \sum_{j=0}^{q-1} c_j\,\delta^{(j)}(x) \quad (14)$$

where $c_0,\ldots..,c_{q-1}$ are arbitrary constants.

4) If $\exists j_* \in \mathbb{Z}_+\setminus\{s\}$ such that, $B - A(j_*+q+1) = 0$ and $\frac{B}{A} < 1$,

$$y(x) = \frac{(-1)^q s!}{(s+q)![B-A(s+q+1)]}\delta^{(q+s)}(x) + c_{j_*+q}\delta^{(j_*+q)}(x) + \sum_{j=0}^{q-1} c_j\,\delta^{(j)}(x) + k_1 x^{-\frac{B}{A}}\theta(x) + k_2|x|^{-\frac{B}{A}}\theta(-x), \quad (15)$$

where $c_0,\ldots..,c_{q-1},c_{j_*+q},k_1,k_2$ are arbitrary constants.

Theorem 3.4 Let $A.B \neq 0, p \in \mathbb{N}, q \in \mathbb{Z}_+$ and accomplished the condition $q < p-1$. Then the general solution of the equation (1) in the space of generalized functions $(S_0^{\beta})'$ has the following form:

1) If $\frac{B}{A} < 0$ and $q-p$ odd number,

$$y(x) = k_1 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(x) + k_2 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(-x) + \sum_{j=0}^{q-1} c_j\delta^{(j)}(x) + \frac{(-1)^q s!}{A(s+q)!}\delta^{(s+q)}(x) + \frac{(-1)^q s!}{A(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}\left(\frac{A}{B}\right)^{l}(-1)^{l(p-1-q)} \times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\,\delta^{(m+q)}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\sum_{l=1}^{\infty}\left(\frac{B}{A}(-1)^{p-q-1}\right)^{l} \cdot \prod_{s=0}^{l-1}\frac{[m+q+s(p-1-q)]!}{[m+p+s(p-1-q)]!}\delta^{(m+q+l(p-1-q))}(x), \quad (16)$$

where $k_1, k_2, c_0, \ldots\ldots, c_{q-1}, C_{m+q}, m = 0, 1, 2, \ldots\ldots, p-q-2$ are arbitrary constants.

2) If $\frac{B}{A} < 0$ and $q - p$ even number,

$$y(x) = k_1 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + \frac{(-1)^q s!}{B(s+q)!}\delta^{(s+q)}(x) +$$

$$\frac{(-1)^q s!}{B(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(\frac{A}{B})^l(-1)^{l(p-1-q)} \times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\,\delta^{(m+q)}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\sum_{l=1}^{\infty}\left(\frac{B}{A}(-1)^{p-q-1}\right)^l \cdot \prod_{s=0}^{l-1}\frac{[m+q+s(p-1-q)]!}{[m+p+s(p-1-q)]!}\delta^{(m+q+l(p-1-q))}(x), \quad (17)$$

where $k_1, c_0, \ldots\ldots, c_{q-1}, C_{m+q}$, $m = 0, 1, 2, \ldots\ldots, p-q-2$ are arbitrary constants , $\theta(x)$ – Heaviside step function.

3) If $\frac{B}{A} > 0$ and $q - p$ odd number,

$$y(x) = \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + \frac{(-1)^q s!}{B(s+q)!}\delta^{(s+q)}(x) +$$

$$\frac{(-1)^q s!}{B(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(\frac{A}{B})^l(-1)^{l(p-1-q)} \times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\,\delta^{(m+q)}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\sum_{l=1}^{\infty}\left(\frac{B}{A}(-1)^{p-q-1}\right)^l \cdot \prod_{s=0}^{l-1}\frac{[m+q+s(p-1-q)]!}{[m+p+s(p-1-q)]!}\delta^{(m+q+l(p-1-q))}(x), \quad (18)$$

where $c_0, \ldots\ldots, c_{q-1}, C_{m+q}, m = 0,1,2, \ldots\ldots, p-q-2$ are arbitrary constants.

4) If $\frac{B}{A} > 0$ and $q - p$ even number,

$$y(x) = k_2 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(-x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + \frac{(-1)^q s!}{B(s+q)!}\delta^{(s+q)}(x) +$$

$$\frac{(-1)^q s!}{B(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(\frac{A}{B})^l(-1)^{l(p-1-q)} \times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\,\delta^{(m+q)}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\sum_{l=1}^{\infty}\left(\frac{B}{A}(-1)^{p-q-1}\right)^l \cdot \prod_{s=0}^{l-1}\frac{[m+q+s(p-1-q)]!}{[m+p+s(p-1-q)]!}\delta^{(m+q+l(p-1-q))}(x), \quad (19)$$

where $k_2, c_0, \ldots\ldots, c_{q-1}, C_{m+q}$; $m = 0, 1, 2, \ldots\ldots, p-q-2$ are arbitrary constants, $\theta(-x) -$ Heaviside step function.

Before concluding this work, let us make this following important remark.

Remark.

From this achieved work, it is clear that the expressions of the obtained solutions are quite huge in some cases and, to verify the accomplishment of the equality

$$Ax^p y'(x) + Bx^q y(x) = 0,$$

one need to realize serious computations by replacing $y(x) \in (S_0^\beta)'$ into the considered equation.

For example in the case when $A.B \neq 0$, $p \in \mathbb{N}, q \in \mathbb{Z}_+$ and accomplished the condition $q < p - 1$, if $\frac{B}{A} < 0$ and $q - p$ odd number, one need to verifiy the realization of the following relationship:

$$\begin{cases} Ax^p y'(x) + Bx^q y(x) = 0 \Leftrightarrow \left(Ax^p y'(x) + Bx^q y(x), \varphi(x)\right) \\ \qquad = \left(0, \varphi(x)\right) \\ \qquad = 0; \quad \forall\, y(x) \in (S_0^\beta)' \forall \varphi(x) \in S_0^\beta . \end{cases} \quad (20)$$

where the generalized-function solution $y(x)$ is defined in this case by the formula:

$$y(x) = k_1 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(x) + k_2 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(-x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + \frac{(-1)^q s!}{A(s+q)!}\delta^{(s+q)}(x) +$$

$$\frac{(-1)^q s!}{A(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(\frac{A}{B})^l(-1)^{l(p-1-q)} \times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\,\delta^{(m+q)}(x) + \sum_{m=0}^{q-p-2} C_{m+q}\sum_{l=1}^{\infty}\left(\frac{B}{A}(-1)^{p-q-1}\right)^l \cdot \prod_{s=0}^{l-1}\frac{[m+q+s(p-1-q)]!}{[m+p+s(p-1-q)]!}\delta^{(m+q+l(p-1-q))}(x).$$

One should compute $y'(x)$the first derivative of the generalized-function solution $y(x)$ and replace it as needed with the expression of$y(x)$ in equation (1) before calculating to reach the awaited the result which is zero. Now let us move to the conclusion and, after what we reach the recommendations closing this work.

CONCLUSION

Summarizing what has been done, we can say that we have completely achieved the investigation of the question related to the appearence of other kind of solutions of the homogeneous differential equation (1) in the space$(S_0^\beta)'$. Similarly to previous approaches undertook in our researches, we have substituted the form of the particular solution as linear combination of Dirac delta function and it derivatives $y(x) = \sum_{m=0}^{\infty} C_j\, \delta^{(j)}(x)$,with unknown coefficients C_j

into the equation (1). This step leads us to an infinity linear homogeneous algebraic system from which, we found the unknown coefficients C_j in each of case and, we formalize in theorems 3.1, 3.2, 3.3 and 3.4 the general form of the solution of the considered equation, taking into account our results obtained in $[25]$ with dependance of the relationship between the parameters *A, B, p* and *q*.

RECOMMENDATIONS

Accordingly to what has been done, it is challenging to try to set the same problem when considering a more general situation of an homogeneous equation of *l*-order in the space $\left(S_0^{\beta}\right)'$ i.e $\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = 0$, with the parameters $k_i \in \mathbb{Z}_+, k_i \geq i, i = 0,1,\ldots\ldots,l;$ and $k_i = k_0 + i$; for$i = 0,1,\ldots\ldots,l - 1$ with$k_l \neq k_0 + l$. The previous is not easy problem that is why it would be wise to begin with a more easy case as we have already investigated the question of the existence of the distributional solutions of the equation $\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = 0$ in the Euler case situation in the space *K'* , by seeking in a brief future the existence of other kind of solutions appearing in the space $\left(S_0^{\beta}\right)$'as done in this work.

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